Lecture Notes in Mathematics

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Tammo tom Dieck

Transformation Groups and Representation Theory

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Transformation Groups and Representation Theory

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Preface

These are extended lecture notes for a course on transformation groups which I gave at the Mathematical Institute at Göttingen during the summer term 1978.

The purpose of these notes is to give an introduction to that part of the theory of transformation groups which centers around the Burnside ring and the topology of group representations. It is assumed that the reader is acquainted with the basic material in algebraic topology, representation theory, and transformation groups. Nevertheless we have presented some elementary topics in detail.

Section 11 contains joint work with Henning Hauschild.

My thanks are due to Christian Okonek who read part of the manuscript and made many valuable suggestions and to Margret Rose Schneider who typed the manuscript.

Contents

1. The Burnside ring of finite G-sets.

In this section let G denote a finite group. In order to motivate some of the subsequent investigations we give an introduction to the Burnside ring of a finite group. Later we generalize this to compact Lie groups by geometric methods which in case of a finite group are not always suitable for the applications of the Burnside ring in representation theory. The material in this section is mainly due to Andreas Dress.

1.1. Finite G-sets.

A finite G-set S is a finite set together with a left action of G on this set. A finite G-set is the disjoint union of its orbits. The orbits are transitive G-sets and are G-isomorphic to homogeneous G-sets $G/H = \{gH \mid g \in G\}$. The G-sets G/H and G/K are isomorphic if and only if H is conjugate to K in G. The set of G-isomorphism classes of finite G-sets becomes a commutative semi-ring $A^+(G)$ with identity with addition induced by disjoint union and multiplication induced by cartesian product with diagonal action. The non-triviality of the multiplication arises from the decomposition of $G/H \times G/K$ into orbits. These orbits correspond to the double cosets HgK, $g \in G$, which can be identified with the orbit space of G/K under the left H-action. This correspondence can be described as follows: If X is an H-space the H-orbits of X correspond to the G-orbits of $G \times_H X$. If moreover X is a G-space then we have the G-isomorphism $G/H \times X \longrightarrow G \times_H X : (g,x) \longmapsto (g,g^{-1}x)$. We apply this to X = G/K. Explicitely, the double coset HgK corresponds to the orbit through (1,g).

1.2. The Burnside ring A(G).

The Grothendieck ring constructed from the semi-ring $A^+(G)$ is denoted A(G) and will be called the Burnside ring of G. If S is a finite G-set let $[S]$ or S be its image in $A(G)$. Additively, $A(G)$ is the free abelian group on isomorphism classes of transitive G-sets. Equivalently, an additive Z-basis is given by the $[G/H]$ where (H) runs through the set $C(G)$ of conjugacy classes of subgroups of G. The multiplication comes from the decomposition of $G/H \times G/K$ into orbits. The ring $A(G)$ is commutative with unit $[G/G]$.

Example 1.2.1.

Let G be abelian. Then, since generally the isotropy group of $G/H \times G/K$ at (g_1H, g_2K) is $g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$, all isotropy groups are $H \cap K$ in the abelian case. Therefore $[G/H] \cdot [G/K] = a[G/H \cap K]$ where $a \in Z$ is obtained by counting the number of elements on both sides. In particular $[G/H]^2 = |G/H| \, [G/H]$, where $|S|$ is the cardinality of S. We see that for abelian G the $[G/H]$ are almost idempotent.

If $H < G$ and S,T are finite G-sets then we have for the cardinality of the H-fixed point sets $|S^H + T^H| = |S^H| + |T^H|$ and $|(S \times T)^H| = |S^H| \, |T^H|$. Hence $S \longmapsto |S^H|$ extends to a ring homomorphism

$$\varphi_H : A(G) \longrightarrow Z .$$

Conjugate subgroups give the same homomorphism so that we have one φ_H for each $(H) \in C(G)$. We let

$$\varphi = (\varphi_H) : A(G) \longrightarrow \prod_{(H) \in C(G)} Z$$

be the product of the φ_H.

Proposition 1.2.2.

φ is an injective ring homomorphism.

Proof.

By definition φ is a ring homomorphism. Suppose $x \neq 0$ is in the kernel of φ. We write x in terms of the basis $x = \sum a_H [G/H]$. We have a partial ordering on the $[G/H]$, namely $[G/H] \leq [G/K]$ if and only if H is subconjugate to K. Let $[G/H]$ be maximal among the basis elements with $a_H \neq 0$. Then $G/K^H \neq \emptyset$ implies $[G/H] \leq [G/K]$. Hence $0 = \varphi_H x =$
$= a_H [G/H^H] = a_H |NH/H| \neq 0$, a contradiction.

Since φ is an injection of a subgroup of maximal rank the cokernel is a finite group. We want to compute its order. We consider the diagram of injective ring homomorphisms

$$\begin{array}{ccc} A(G) & \xrightarrow{\varphi} & \prod Z \\ \downarrow & & \downarrow \\ A(G) \otimes Q & \xrightarrow{\varphi_Q} & \prod Q \end{array}$$

where the lower φ_Q is the rational extension of the upper φ.

Recall that $WH = NH/H$ acts freely on G/H as the group of G-automorphisms: The action is given by $WH \times G/H \longrightarrow G/H : (wH, gH) \longmapsto gw^{-1}H$. Hence it acts freely on any fixed point set G/H^K. In particular $|G/H^K|$ is divisible by $|WH|$. Therefore $\varphi_Q([G/H] \otimes |WH|^{-1})$ is contained $\prod Z$.

Proposition 1.2.3.

The elements $\varphi_Q([G/H] \otimes |WH|^{-1}) =: x_H$ form a Z-basis of $\prod Z$. The order of cokernel φ is $\prod_{(H) \in C(G)} |WH|$.

Proof.

The first assertion implies the second one. We view elements in $\prod Z$ as row vectors. Then the x_H form (suitably ordered) a triangular matrix with one's on the diagonal. Hence they must be a basis.

Remark 1.2.4.

The homomorphism φ may be discovered from the ring structure of A(G) as follows. An element $x \in A(G)$ is a non-zerodivisor if and only if φx has no zero component. Therefore $A(G) \otimes Q$ is the total quotient ring of A(G) (i.e. all non-zero-divisors made invertible). If $x \in A(G) \otimes Q$ is integral over A(G) then the components of $\varphi_Q x$ are integral over Z hence integers. Conversely $\prod Z$ is integral over $\varphi A(G)$, e. g. because $\prod Z$ is generated by idempotent elements which are integral over any subring. Hence φ may be identified with the inclusion of A(G) into the integral closure in its total quotient ring. (For the notion of integral ring extension see Lang [107], Chapter IX; Bourbaki [33], Ch. 5.)

1.3. Congruences between fixed point numbers.

We have seen in 1.2. that $\varphi A(G)$ is a subgroup of maximal rank in $\prod Z$. How can we describe its image? If $G = Z/pZ$ is the cyclic group of prime order p then $|S| \equiv |S^G| \bmod p$ because the orbits of $S \setminus S^G$ have cardinality p. Hence this congruence gives a condition for elements to be in the image of φ. The reader can easily check that this is the only condition, for $G = Z/pZ$. We generalize such congruences.

Let S be a finite G-set and let V(S) be the complex vector space spanned by the elements of S. The G-action on the basis S of V(S) induces a linear action on V(S). The resulting G-module V(S) is called the permutation representation associated to S. The character of V(S) is a function on G; it will be denoted with the same symbol. The orthogonality relations for characters say in particular that for any complex G-module V the number $|G|^{-1} \sum_{g \in G} V(g)$ is the dimension of V^G. Hence

$$(1.3.1) \qquad \sum_{g \in G} V(S)(g) \equiv 0 \mod |G| .$$

Now note that

$$V(S)(g) = \mathrm{Trace}(1_g : V(S) \longrightarrow V(S) : v \longmapsto gv) = |S^g|$$

(look at the matrix of 1_g with respect to the basis S). Therefore 1.3.1 can be rewritten

$$(1.3.2) \qquad \sum_{g \in G} \varphi_{\langle g \rangle}(x) \equiv 0 \mod |G|$$

for any $x \in A(G)$, where $\langle g \rangle$ denotes the cyclic group generated by g. If H is a cyclic subgroup of G the number of elements g with $\langle g \rangle$ conjugate to H is

$$|H^*|\,|G/NH|$$

where H^* is the set of generators of H and $|G/NH|$ is the number of groups conjugate to H. Therefore (1.3.2) can be rewritten

$$(1.3.3) \qquad \sum_{(H)\ \text{cyclic}} |H^*|\,|G/NH|\ \varphi_H(x) \equiv 0 \mod |G|$$

where now the summation is taken over conjugacy classes of cyclic subgroups of G.

We now apply the same argument to $V(S^H)$ considered as NH/H-module and obtain

$$\sum_{(K)} |NK/NH \cap NK| \, |K/H^{*}| \quad \varphi_K(x) \equiv 0 \mod |NH/H|$$

where we sum over NH-conjugacy classes K such that H is normal in K and K/H is cyclic. This may also be written in the form

(1.3.4) $$\sum_{(K)} n(H,K) \quad \varphi_K(x) \equiv 0 \mod |NH/H|$$

where the n(H,K) are certain integers with n(H,H) = 1 and the sum is over the G-conjugacy classes (K) such that H is normal in K and K/H is cyclic.

For the next Proposition we view elements of $\prod Z$ as functions $C(G) \longrightarrow Z$.

Proposition 1.3.5.

The congruences 1.3.4 are a complete set of congruences for image φ, i. e. $x \in \prod Z$ is contained in the image of φ if and only if

$$\sum_{(K)} n(H,K) \; x(K) \equiv 0 \mod |NH/H|$$

for all $(H) \in C(G)$.

Proof.

We have already seen that the elements in the image of φ satisfy these congruences. The congruences 1.3.6 are independent because they are given by a triangular matrix with one's on the diagonal. Hence they describe a subgroup A of index $\prod |NH/H|$. By Proposition 1.2.3 therefore $A = \text{im } \varphi$.

Remark 1.3.7.

A slightly different set of congruences is obtained if one considers $V(S^H)$ as N_pH/H-module where N_pH/H is a Sylow p-group of NH/H. This yields a set of p-primary congruences which may be used instead of 1.3.4. These congruences are useful when localizations of A(G) are considered; e. g. for $A(G)_{(p)}$, the Burnside ring localized at p, only p-primary congruences are valid.

1.4. Idempotent elements.

Idempotent elements in $\prod Z$ are the functions with values 0 and 1. We use 1.3 to see when such functions come from A(G). We consider A(G) as subring of $\prod Z$ via φ .

A subgroup H of G is called perfect if it is equal to its commutator subgroup. Each $H < G$ has a smallest normal subgroup H_s such that H/H_s is solvable. One has $(H_s)_s = H_s$. A subgroup H is perfect if and only if $H = H_s$. Let P(G) be the subset of C(G) represented by perfect subgroups.

Proposition 1.4.1.

An idempotent $e \in \prod Z$ is contained in A(G) if and only if for all $(H) \in C(G)$ the equality $e(H) = e(H_s)$ holds.

Proof.

Suppose $e \in A(G)$. Then e satisfies 1.3.6. Given $K < G$. Choose $K_s = K^n \triangleleft K^{n-1} \triangleleft \dots \triangleleft K^o = K$ such that K^i/K^{i+1} is cyclic of prime order p(i). Then by 1.3.6 applied to the group K^{i+1} we have $e(K^i) \equiv e(K^{i+1})$ mod p(i). Since the values of e are 0 or 1 we must have $e(K^i) = e(K^{i+1})$ and therefore $e(K_s) = e(K)$. Conversely assume that $e(K_s) = e(K)$ for all K. Then we must have $e(H) = e(K)$ for all $H \triangleleft K$ with K/H cyclic so that e satisfies the congruences 1.3.6.

Corollary 1.4.2.

The set of indecomposable idempotents of A(G) corresponds bijectively to P(G). In particular G is solvable if and only if 0 and 1 are the only idempotents in A(G).

Remark 1.4.3.

Let $P \subset Z$ be a set of prime numbers. Let $A(G)_P$ be the localization of A(G) at P, i. e. the primes not in P are made invertible. Then one can show as in the proof of Proposition 1.4.1 that the idempotents of $A(G)_P$ are the functions e with $e(H) = e(H_P)$ where H_P is the smallest normal subgroup of H such that H/H_P is solvable of order involving only primes in P.

1.5. Units.

If A is a commutative ring we let A^* be the multiplicative group of its units.

Let $e \in A$ be an idempotent. Then $1-2e = u$ is a unit. Conversely it can happen that for a unit u the element $(1-u)/2 = e$ is contained in A. Then e is an idempotent, because $(1-u)^2 = 2(1-u)$ for any unit u. In case of the Burnside ring $(1-u)/2$ is contained in $\prod Z$ but not in general in A(G) as we shall see in a moment. But if G has odd order then coker φ is odd and hence $1-u \in A(G)$ and $(1-u)/2 \in \prod Z$ implies $(1-u)/2 \in A(G)$. Since a non-solvable group has non-trivial idempotents, by 1.4.2, we obtain

Proposition 1.5.1.

If G is non-solvable then $A(G)^* \neq \{\pm 1\}$. If G is solvable of odd order then $A(G)^* = \{\pm 1\}$.

Let H be a subgroup of index 2 in G. Then $H \triangleleft G$, $[G/H]^2 = 2\,[G/H]$

and therefore $u(H) := 1 - [G/H] \in A(G)^*$. Note that $(1-u(H))/2$ is not in $A(G)$. The subgroups of index 2 are precisely the kernels of non-trivial homomorphisms $G \longrightarrow Z/2Z$. Hence we obtain an injective map $j : Hom(G,Z/2Z) \longrightarrow A(G)^*$ given by $j(f) = 1-G/ker(f)$. The image of j is in general not a subgroup.

Problem 1.5.2.

Determine the structure of $A(G)^*$ in terms of the structure of G. (Of course one knows by the famous theorem of Feit - Thompson that groups of odd order are solvable. Therefore the 2-primary structure of G is relevant. In particular $A(G)^*$ for 2-groups would be interesting. (See also the next remark.)

Remark 1.5.3.

We shall prove later by geometric methods that for a real representation V the function $(H) \longmapsto (-1)^{\dim V^H}$ is contained in $A(G)$. This function is then a unit in $A(G)$. It would be interesting to see units which are not of this form (2-groups?).

1.6. Prime ideals.

Since ΠZ is integral over $A(G)$ by the "going-up theorem" of Cohen-Seidenberg (see Atiyah-Mac Donald [11] , p. 62) every prime ideal of $A(G)$ comes from ΠZ hence has the form

$$q(H,p) := \{ x \in A(G) \mid \varphi_H(x) \equiv 0 \mod p \}$$

for a subgroup H of G and a prime ideal (p) of Z. The elementary proof of Dress [79] for this fact shall be given later (section 5) in the slightly more general context of compact Lie groups. The prime ideals $q(H,o)$ are minimal; the ideals $q(H,p)$, $p \neq 0$, are maximal with residue field Z/pZ. If $q(H,p) = q(K,q)$ then $p = q$ and

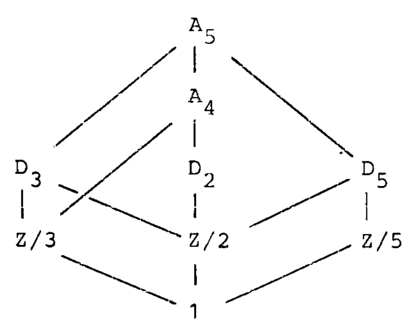
(i) $(H) = (K)$ if $p = 0$

(ii) $(H_p) = (K_p)$ if $p \neq 0$.

Here H_p is the smallest normal subgroup of H such that H/H_p is a p-group. If q is a prime ideal of $A(G)$ then there exists a unique minimal (H) such that $[G/H] \notin q$. Moreover for this H one has $q = q(H,p)$ where p is the characteristic of the ring $A(G)/q$. Finally this (H) is the maximal (H) for which $q = q(H,p)$. All this is proved in Dress [79] and will later be proved for compact Lie groups.

1.7. An example: The alternating group A_5.

The diagram of conjugacy classes of subgroups of A_5 is



Here D_n is the dihedral group of order $2n$. The groups A_5, A_4, D_5, D_3 are their own normalizers while $N(Z/n) = D_n$ and $N(D_2) = A_4$. $A(A_5)$ is the set of functions $z : C(G) \longrightarrow Z$ satisfying

(i) $z(H)$ arbitrary for $H = A_5, A_4, D_5, D_3$.

(ii) $z(Z/n) \equiv z(D_n) \bmod 2$ for $n = 3,5$.

(iii) $z(D_2) \equiv z(A_4) \bmod 3$.

(iv) $z(1) + 20z(Z/3) + 15z(Z/2) + 24z(Z/5) \equiv 0 \bmod 60$.

The ring $A(A_5)$ contains the following units:

1	Z/2	Z/3	Z/5	D_2	D_3	D_5	A_4	A_5
a	a	a	a	b	c	d	b	e

Here a,b,c,d,e $\in \{1,-1\}$ and the second line gives the value of the function $u : C(G) \longrightarrow Z$ at the element indicated in the first line. The congruences (i) - (iv) show that there are no conditions for a unit u at A_5, A_4, D_3. From (iii) we obtain $u(D_2) = u(A_4)$. Considering (iv) mod 3, mod 4, and mod 5 we obtain

$$u(1) = u(Z/2) = u(Z/3) = u(Z/5).$$

The subgroups 1 and A_5 are perfect. Therefore $A(A_5)$ contains the idempotents 0,1,e,1-e where $\varphi_{A_5}(e) = 1$, $\varphi_H(e) = 0$ for $H \neq A_5$.

1.8. Comments.

The Burnside ring was introduced by Dress [79] where also the prime ideal spectrum was determined. The Burnside ring plays an important role in the axiomatic representation theory (Green [38] , Dress [80]) in particular in the general theory of induction theorems (Dress [80]). The Burnside ring, as a functor, is universal among the Mackey functors of Dress, see the cited references.

We shall demonstrate in these lectures the topological significance of the Burnside ring. At this point we only mention that a finite simplicial complex with simplicial G-action is a combinatorial object built from finite G-sets. So one expects some basic invariants of simplicial G-complexes to lie in the Burnside ring, e.g. the "Euler-Characteristic": the alternating sum $\sum (-1)^i S_i$ of the G-sets S_i of i-simplices.

The Burnside ring codifies in a convenient frame-work some basic properties of the lattice of subgroups of a given group. Given G, the

G-transformation groups are governed by the internal relations of the Burnside ring. This influence of the Burnside ring is more transparent when we have shown that the ring is isomorphic to equivariant stable homotopy of sphere in dimension zero (Segal [145]) so that in particular stable equivariant homotopy groups are modules over the Burnside ring.

The description of the Burnside ring using congruences among cardinalities of fixed point sets is based on an oral communication by Dress. These congruences are generalized in tom Dieck-Petrie [69] where also various geometrical applications are given.

1.9. Exercises.

1. Let G and H be finite groups whose orders are relatively prime. Show that

$$A(G \times H) \cong A(G) \otimes A(H)$$

2. For $i \not\equiv 0 \bmod p$ let

$$M(i) = \{(a,b) \mid ai \equiv b \mod p\} \subset Z \times Z.$$

Show that M(i) is a projective module over A(Z/pZ). Classify projective modules over A(Z/pZ).

3. Show that G is perfect if and only if A(G) contains the idempotent e such that

$$\varphi_H e = 0 \quad \text{for} \quad H \neq G, \quad \varphi_G e = 1.$$

4. Let G be a p-group of order p^n (p a prime). Let $m \subset A(G)$ be the ideal

$$m = \{x \mid \varphi_{\{1\}} x \equiv 0 \mod p\}.$$

Show that $m^{n+1} \subset p\,A(G)$. (In particular: The p-adic and the m-adic topologies coincide.)

5. Let G be a 2-group and let $|A(G)^*| = 2^n$. Show that n is not greater than the number of conjugacy classes (H) such that $|NH/H| = 2$.

2. The J-homomorphism and quadratic forms.

Having defined the Burnside ring of finite G-sets in the previous chapter we go on to study finite G-sets which arise from G-modules over finite fields and G-invariant quadratic forms on such modules. This will later be used to study permutation representations. In this chapter G will always denote a finite group.

2.1. The J-homomorphism.

We consider torsion G-modules M, i. e. finite abelian groups M together with a left G-action by group automorphisms. Forgetting the group structure on M yields a finite G-set and therefore an element J(M) in the Burnside ring A(G). Since $\varphi_H J(M) = |M^H|$ we have

$$J(M \oplus N) = J(M)\, J(N) \qquad (2.1.1)$$

for two torsion G-modules M and N. But J(M) is in general not a unit in A(G) so that J does not immediately extend to a homomorphism from a suitable Grothendieck group. On the category of torsion modules with torsion prime to $|G|$ taking H-fixed points is an exact functor so that $J(M) = J(N)\, J(P)$ for an exact sequence $0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0$ of such modules.

Proposition 2.1.2.

Let M be a torsion G-module with $q = |M|$ prime to $|G|$. Then $J(M) \in A(G)[q^{-1}]$ (i.e. q made invertible) is a unit.

Proof.

Using φ of 1.2.2 we see that $\varphi\, J(M)$ is certainly a unit in $\prod Z[q^{-1}]$. We have to show: the inverse is contained in $A(G)[q^{-1}]$. Note that by

1.2.3 the cokernel of $\varphi[q^{-1}]$ is a finite group because q is prime to $|G|$. The next algebraic lemma implies the result.

Lemma 2.1.3.

Let R be a subring of the commutative ring S. Assume that $R \subset S$ is an integral extension (e.g. S/R is a finite group). Then $R^* = R \cap S^*$.

Proof.

Clearly $R^* \subset S^*$. Given $x \in R \cap S^*$. Suppose $y \in S$ satisfies $xy = 1$. Since $S \supset R$ is integral we have $y^n + a_1 y^{n-1} + \ldots + a_n = 0$ for suitable $a_i \in R$. Multiplying this equation by x^{n-1} we obtain $y + a_1 + \ldots + a_n x^{n-1} = 0$, hence $y \in R$.

Let $T_q(G)$ be the Grothendieck group with respect to exact sequences of q-torsion G-modules. Let R(G;F) be the Grothendieck group of finitely generated FG-modules, F a field. Then 2.1.2 implies

Proposition 2.1.4.

Let q be prime to $|G|$. The assignment $M \mapsto J(M)$ induces a homomorphism $J : T_q(G) \longrightarrow A(G)[q^{-1}]^*$. If F is a finite field of characteristic q then we obtain a homomorphism $J : R(G;F) \longrightarrow A(G)[q^{-1}]^*$.

We call this homomorphism the J-*homomorphism*. The relation to the J-homomorphism of algebraic topology will become clear later.

2.2. Quadratic forms on torsion groups. Gauß sums.

Let M be a finite abelian group.

Definition 2.2.1.

A *quadratic form on* M is a map $q : M \longrightarrow Q/Z$ such that

i) q is quadratic, i. e. $q(am) = a^2 q(m)$ for $a \in Z$ and $m \in M$.

ii) the map $b : M \times M \longrightarrow Q/Z$,

$b(m,n) = q(m+n) - q(m) - q(n)$ is biadditive.

If moreover M is a ZG-module we call q G-<u>invariant</u> if for $g \in G$ and $m \in M$ the relation $q(gm) = q(m)$ holds. The form is called <u>non-degenerate</u> if $b^* : M \longrightarrow Hom(M,Q/Z) : m \longmapsto b(m,-)$ is an isomorphism.

We shall only consider non-degenerate forms. Let $e : Q/Z \longrightarrow \mathbb{C}^*$ be the standard character $e(x \bmod Z) = \exp(2\pi ix)$.

<u>Definition 2.2.2.</u>

Let (M,q) be a quadratic torsion form. The <u>associated</u> (<u>quadratic</u>) <u>Gauß sum</u> is

$$G(M,q) = \sum_{m \in M} eq(m).$$

(We use the letter G despite of its use for groups.)

We now list some formal properties of Gauß sums. If (M_1,q_1) and (M_2,q_2) are quadratic torsion forms we have the <u>orthogonal sum</u>

$$(M_1,q_1) \perp (M_2,q_2) =: (M,q)$$

which is $(M_1 \oplus M_2, q)$ with

$$q(m_1,m_2) = q_1(m_1) + q_2(m_2).$$

Obviously one has

(2.2.3) $$G(M,q) = G(M_1,q_1)\, G(M_2,q_2).$$

Definition 2.2.4.

A quadratic torsion form (M,q) is called split or metabolic if there exists a subgroup $N \subset M$ such that for all $n \in N$ $q(n) = 0$ and moreover $N^{\perp} := \{n \mid b(n,N) = 0\}$ equals N. We then call N a metabolizer of (M,q).

Proposition 2.2.5.

If (M,q) is split with metabolizer N then $G(M,q) = |N|$.

Proof.

Since q is non-degenerate the map

$$M \longrightarrow \mathrm{Hom}(M,Q/Z) \longrightarrow \mathrm{Hom}(N,Q/Z)$$

is surjective with kernel $N^{\perp}$. By assumption $N = N^{\perp}$. The induced map $\bar{b} : N \times M/N \longrightarrow Q/Z$ is non-degenerate. Therefore $|N| = |M/N|$, $|M| = |N|^2$. For $m \in M$ we have

$$\sum_{n \in N} eq(m+n) = eq(m) \sum_{n \in N} eb(m,n) .$$

If $m \notin N$ then $n \longmapsto eb(m,n)$ is a non-trivial character of N. The sum above is therefore zero in this case. There remains the sum for $m = 0$ which is equal to $|N|$.

If (M,q) is torsion form, then $(M,q) \perp (M,-q)$ is always split, with metabolizer the diagonal of $M \oplus M$. On the set $KQ^{+}(Q,Z)$ of isomorphism classes of quadratic torsion forms one has the relation of Witt equivalence: $(M_1,q_1) \sim (M_2,q_2)$ if and only if there exist split forms (V_i,r_i) such that $(M_1,q_1) \perp (V_1,q_1) \cong (M_2,q_2) \perp (V_2,q_2)$. The set of Witt equivalence classes $WQ(Q/Z)$ becomes an abelian group, the group structure being induced from orthogonal sum. From 2.2.5 we see that the assignment $(M,q) \longmapsto G(M,q)/\sqrt{|M|}$ induces a homomorphism

$$(2.2.6) \qquad \gamma : WQ(Q/Z) \longrightarrow \mathbb{C}^* .$$

In particular we have

$$(2.2.7) \qquad |G(M,q)|^2 = |M|$$

for any torsion form and $\gamma(M,q)$ is a root of unity.

For the convenience of the reader we now collect the relevant material about Witt groups. The general reference will be Milnor-Husenmoller [117] . Let W(R) be the Witt ring of symmetric inner product spaces ([117] , p. 14) and WQ(R) the Witt algebra of quadratic forms ([117] , p. 112) for a commutative ring R. If we assign to a quadratic form its associated bilinear form we obtain a homomorphism

$$a : WQ(R) \longrightarrow W(R)$$

which is an isomorphism if 2 is a unit in R. There is an exact sequence ([117] , p. 90)

$$(2.2.8) \qquad 0 \longrightarrow W(Z) \longrightarrow W(Q) \longrightarrow W(Q/Z) \longrightarrow 0$$

where W(Q/Z) is the Witt group of symmetric bilinear forms on torsion groups. Moreover

$$W(Q/Z) \cong \bigoplus_p W(Z[p^{-1}]/Z)$$

because a torsion form is uniquely the orthogonal sum of its restrictions to the p-primary parts. Moreover one has an isomorphism

(2.2.9) $$W(F_p) \cong W(Z[p^{-1}]/Z)$$

viewing a form over the ring $F_p = Z/pZ$ as a torsion form. The ring $W(F_p)$ is computed in [117], p. 87. One has $W(Z) = Z$ by the signature homomorphism and the signature splits $W(Z) \longrightarrow W(Q)$.

In the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & WQ(Z) & \longrightarrow & WQ(Q) & \longrightarrow & WQ(Q/Z) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a(Z)} & & \downarrow{\scriptstyle a(Q)} & & \downarrow{\scriptstyle a(Q/Z)} & & \\
0 & \longrightarrow & W(Z) & \longrightarrow & W(Q) & \longrightarrow & W(Q/Z) & \longrightarrow & 0
\end{array}$$

the map $a(Q)$ is an isomorphism and so is $a(Z[p^{-1}]/Z)$ for p odd. The map $a(Z)$ is injective with cokernel of order 8 ([117] , p.24). The map

$$WQ(Z[2^{-1}]/Z) \longrightarrow W(Z[2^{-1}]/Z)$$

is surjective and the source is isomorphic to $Z/8Z \times Z/2Z$. A torsion form of order 8 in the Witt group is

$$q : Z/2Z \longrightarrow Q/Z$$

$$q(0) = 0, \quad q(1) = \frac{1}{4}.$$

The value $\gamma(Z/2Z,q)$ of 2.2.6 is in this case $\frac{1}{\sqrt{2}}(1+i)$, a primitive 8-th root unity.

From the quoted results one sees already that $\gamma(M,q)$ has order 2^i,

$0 \leq i \leq 3$. For the actual computation of γ see [117], Appendix 4, or Lang [108], IV §3.

We now study more closely the case of quadratic forms on F_p-modules (alias torsion form). We assume that p is an odd prime.

If (M,q) is given then for $a \in F_p$, $a \neq 0$

(2.2.10) $$q^{-1}(a^2 b) = aq^{-1}(b)$$

and the sets $q^{-1}(a^2 b)$ and $q^{-1}(b)$ have the same cardinality. Therefore

(2.2.11) $$G(M,q) = \Sigma_{b \bmod p}\, q^{-1}(b)\ \exp(2\pi ib/p)$$

$$= P + Q\alpha + N\beta$$

where

$Q = q^{-1}(b)$ for any non-zero square b in F_p

$N = q^{-1}(c)$ for any non-square c in F_p

$P = q^{-1}(0)$

and

$$-1-\beta = \alpha = \Sigma \exp(2\pi ib/p)$$

summed over the non-zero squares in F_p. We write 2.2.11 as

(2.2.12) $$G(M,q) = P - N + (Q-N)\alpha ,$$

and we are going to compute P - N and Q - N.

We use the following notations:

$$1 + 2\alpha = g = \sum_{a \bmod p} \exp(2\pi i a^2/p)$$

$(\frac{x}{p})$ Legendre symbol

$D(q) \in F_p/F_p^2$ determinant of the form q.

<u>Proposition 2.2.13.</u>

<u>Let</u> (M,q) <u>be a form with</u> $|M| = p^n$. <u>Then</u>

$$G(M,q) = (\frac{D(q)}{p})\ g^n\ .$$

<u>Proof.</u>

Both expressions behave multiplicatively with respect to orthogonal sum. A form over F_p, p odd, is an orthogonal sum of one-dimensional forms. Therefore it suffices to consider the case n = 1. But then the equality is a simple calculation (see Lang [108], QS 1 on p. 85).

From 2.2.12 and 2.2.13 we obtain

$$P - N + (Q-N)(\tfrac{1}{2}(g-1)) = (\frac{D(q)}{p})g^n \qquad (2.2.14)$$

where P also denotes the cardinality of the set P, etc. We now use the fact that the absolute value of g is $\sqrt{p}$. Comparing coefficients gives

<u>Proposition 2.2.15.</u>

<u>If</u> n = 2k, <u>then</u> $Q - N = 0$ <u>and</u> $P - N = (\frac{D(q)}{p})g^{2k}$.

<u>If</u> n = 2k+1, <u>then</u> $2(P-N) = Q-N$ <u>and</u> $P-N = (\frac{D(q)}{p})g^{2k}$.

Remark.

Using $P + \frac{1}{2}(p-1)Q + \frac{1}{2}(p-1)N = p^n$ and 2.2.15 one can solve for P,Q, and N thus obtaining the number of solutions of $q(x) = b$.

Finally we recall the elementary computation (Lang [108] , p. 77)

$$(2.2.16) \qquad g^2 = (\frac{-1}{p})p.$$

2.3. The quadratic J-homomorphism.

We use equivariant Gauß sums to describe certain refinements of the construction in 2.1.

Let M be a ZG-module which is finite as an abelian group and let (M,q) be a G-invariant quadratic form on M as in 2.2. Since $q: M \longrightarrow Q/Z$ is G-invariant the sets

$$q^{-1}(x), \quad x \in Q/Z$$

are finite G-sets. We consider the equivariant Gauß sum

$$(2.3.1) \qquad G(M,q) = \sum_{x \in Q/Z} q^{-1}(x)e(x) .$$

(This is essentially a finite sum). We think of G(M,q) as an element in

$$A(G)[\zeta] = A(G) \otimes_Z Z[\zeta] \subset A(G) \times_Z \mathbb{C}$$

where ζ is a root of unity that generates eqM. For an orthogonal sum we have

(2.3.2) $$G((M_1,q_1) \perp (M_2,q_2)) = G(M_1,q_1)G(M_2,q_2)$$

If we forget the G-action, i. e. put $|q^{-1}(x)| \in Z$ in 2.3.1, then we obtain the Gauß sum G(M,q) of 2.2. Since $b^*: M \longrightarrow Hom(M,Q/Z)$ is an ZG-isomorphism by assumption, q induces on each fixed point set M^H a quadratic form called q^H. Therefore

(2.3.3) $$G(M,q)^H = G(M^H,q^H)$$

with the obvious meanings of the symbols.

As in 2.2.12 we can write

$$G(M,q) = P - N + (Q-N)\alpha$$

where now P,N, and Q are finite G-sets. Here again we work with F_pG-modules, p odd, for simplicity. We describe these G-sets through its fixed point numbers, using 2.2.13. We obtain

Proposition 2.3.4.

Let p be an odd prime and q a G-invariant quadratic form on the F_pG-module M. Then the elements P - N and Q - N of the Burnside ring A(G) have the following fixed point functions:

$$P - N : (H) \longmapsto \left(\frac{D(q^H)}{p}\right) p_*^{[\frac{1}{2}\dim M^H]}$$

$$Q - N : (H) \longmapsto (1-(-1)^{\dim M^H}) \left(\frac{D(q^H)}{p}\right) p_*^{[\frac{1}{2}\dim M^H]}$$

with $p_* = \left(\frac{-1}{p}\right)p$.

Here $[x]$ is the greatest integer m such that $m \leq x$ and dim is the dimension as F_p vector space. (If $M^H = \{o\}$, then $P = 1$, $Q = N = o$.)

This proposition shows that the equivariant Gauß sum of (M;q) only depends on the underlying F_pG-module and the determinant function, i.e. the determinants $D(q^H)$ of all fixed point forms. If $KQ(G;F_p)$ denotes the Grothendieck group of quadratic forms on F_pG-module (with orthogonal sum as addition) we consider the quotient group which only records the isomorphism type of the underlying module and the determinant. We denote this group $RO'(G,F_p)$. We have natural homomorphisms

(2.3.5)
$$r : RO'(G,F_p) \longrightarrow RO(G,F_p)$$
$$d : RO'(G,F_p) \longrightarrow \prod_{(H)} Z^*.$$

Here r associates to the class of (M,q) the underlying F_pG-module M and $RO(G,F_p)$ is simply the image of r in the representation ring $R(G,F_p)$. Hence r is surjective by definition, Moreover d associates to (M,q) the function $(H) \longmapsto (\frac{D(q^H)}{p}) \in Z^* = \{+1,-1\}$. The homomorphism

$$(r,d) : RO'(G,F_p) \longrightarrow RO(G,F_p) \times \prod_{(H)} Z^*$$

is injective, by definition. Hence additively the torsion of $RO'(G,F_p)$ contains only elements of order two and the torsion subgroup is mapped injectively under d.

The assignment

$$(M,q) \longmapsto P - N$$

induces a well-defined map

$$(2.3.6) \qquad JQ : RO'(G,F_p) \longrightarrow A(G)\,[p^{-1}]$$

which is not homomorphic from addition to multiplication. We call JQ the <u>quadratic</u> J-<u>homomorphism</u>.

2.4. Comments.

The construction in 2.1 and 2.3 are taken from Segal [146]. For the localization sequence for Witt groups see Pardon [125], and, in the equivariant case, Dress [81]. The use of equivariant Witt groups in topology is explained in Alexander-Conner-Hamrick [3], where the reader will find many computations. For quadratic forms on torsion see e. g. Wall [164], Brumfiel -Morgan [44], and Alexander-Hamrick-Vick [4]. For 2.2.15 and the remark following it see Siegel [150] p. 344. Proposition 2.3.4 is related to recent work of Tornehave [160] (see Madsen [113]).

2.5. Exercises.

1. Let n be a natural number. Let S be a finite G-set. Let n^S be the function $(H) \longmapsto n^{|S^H|}$. Show that $n^S \in A(G)$.

2. It is seen from 2.3.4 that JQ is not additive. Verify the following formula for the deviation from additivity

$$JQ((M_1,q_1) \perp (M_2,q_2)) = d(M_1,M_2)JQ(M_1,q_1)JQ(M_2,q_2)$$

where

$$d(M_1,M_2) = (1+(p_* -1)\tfrac{1}{4}\ (d(M_1)-1)\ \ (d(M_2)-1))$$

with $d(M) : (H) \longmapsto (-1)^{\dim M^H}$. (Compare 1.5.3)

3. Let F be a field of characteristic not 2 and let G be a group of order prime to char(F). Show that any G-invariant quadratic form over F is an orthogonal sum of indecomposable quadratic modules (M,q). If (M,q) is indecomposable then either M is irreducible and isomorphic to its dual M^* or $M = N \oplus N^*$, $N \neq N^*$, N irreducible, and q is hyperbolic.

4. Extend 2.3.4 to general quadratic forms on torsion groups.

5. Since the signature of $x \in WQ(Z)$ is divisible by 8 the signature homomorphism $WQ(Q) \longrightarrow Z/8Z$ factors over $WQ(Q/Z)$. Compute it! (Compare the formula of Milgram in Milnor-Husemoller [117] , p. 127.)

3. λ-Rings.

We present the theory of special λ-rings. The algebraic material is mainly taken from the paper [14] by Atiyah and Tall. The reader should consult this paper for additional information. The main theorem to be proven here is an exponential isomorphism for p-adic λ-rings which is an algebraic version of the powerful theorem $J'(X) = J''(X)$ in the work of Adams [2] on fibre homotopy equivalence of vector bundles.

3.1. Definitions.

Let R be a commutative ring with identity. A λ-ring structure on R consists of a sequence $\lambda^n : R \longrightarrow R$, $n \in \mathbb{N}$, of maps such that for all $x, y \in R$

(3.1.1)
$$\lambda^o(x) = 1$$
$$\lambda^1(x) = x$$
$$\lambda^n(x+y) = \sum_{r=o}^{n} \lambda^r(x)\ \lambda^{n-r}(y).$$

If t is an indeterminate we define

(3.1.2)
$$\lambda_t(x) = \sum_{n \geqslant o} \lambda^n(x) t^n.$$

Then 3.1.1 shows that

(3.1.3)
$$\lambda_t : R \longrightarrow 1 + R[[t]]^+$$

is a homomorphism from the additive group of R into the multiplicative group $1 + R[[t]]^+$ of formal power series over R with constant term 1.

Exterior powers of modules have formal properties like 3.1.1 and we shall see later how exterior powers give λ-ring structures on certain Grothendieck groups.

A ring R together with a λ-ring structure on it is called a λ-ring. A λ-homomorphism is a ring homomorphism commuting with the λ-operations. We have the notions of λ-ideal and λ-subring.

Some further axioms are needed to insure that the λ-operations behave well with respect to ring multiplication and composition.

Let $x_1,\ldots,x_p$, $y_1,\ldots,y_q$ be indeterminates and let u_i,v_i be the i-th elementary symmetric functions in $x_1,\ldots,x_p$ and $y_1,\ldots,y_q$ respectively. Define polynomials with integer coefficients:

(3.1.4) $P_n(u_1,\ldots,u_n;\ v_1,\ldots,v_n)$ is the coefficient of t^n in $\prod_{i,j}(1+x_iy_jt)$.

(3.1.5) $P_{n,m}(u_1,\ldots,u_{mn})$ is the coefficient of t^n in $\prod_{i_1<\ldots<i_m}(1+x_{i_1}\cdot\ldots\cdot x_{i_m}t)$.

Then P_n is a polynomial of weight n in the u_i and also in the v_i, and $P_{n,m}$ is of weight nm in the u_i. If we assume $p \geq n$, $q \geq n$ in 3.1.4 and $p \geq mn$ in 3.1.5 then non of the variables u_i,v_i involved are zero and the resulting polynomials are independent of p,q.

A λ-ring R is said to be special if in addition to 3.1.1 the following identities hold for $x,y \in R$

(3.1.6)
$$\begin{aligned} \lambda_t(1) &= 1+t \\ \lambda^n(xy) &= P_n(\lambda^1x,\ldots,\lambda^nx;\ \lambda^1y,\ldots,\lambda^ny) \\ \lambda^m(\lambda^n(x)) &= P_{m,n}(\lambda^1x,\ldots,\lambda^{mn}x). \end{aligned}$$

One can motivate 3.1.6 as follows. An element x in a λ-ring is called n-dimensional if $\lambda_t(x)$ is a polynomial of degree n. The ring is called finite-dimensional if every element is a difference of finite dimensional elements. If $x = x_1 + \ldots + x_p$ and $y = y_1 + \ldots + y_q$ in a λ-ring and the x_i, y_i are one-dimensional then

$$\lambda_t(x) = \prod (1+x_i t) = 1 + u_1 t + \ldots + u_p t^p$$

(u_i the i-th elementary function of the x_j as above) and we see that the second identity of 3.1.6 is true for such x,y. If moreover the product of one-dimensional elements is again one-dimensional then the third identity of 3.1.6 is true for $x = \Sigma x_i$. The axioms for a special λ-ring insure that many theorems about λ-rings can be proved by considering just one-dimensional elements. We formalize this remark.

One defines a λ-ring structure on $1+A[[t]]^+$ by:

(3.1.7) "addition" is multiplication of power series.

"multiplication" is given by

$$(1+\Sigma a_n t^n) \circ (1+\Sigma b_n t^n) = 1 + P_n(a_1,\ldots,a_n;b_1,\ldots,b_n)t^n.$$

The " λ-structure" is given by

$$\Lambda^m(1+\Sigma a_n t^n) = 1 + \Sigma P_{n,m}(a_1,\ldots,a_{mn})t^n.$$

Proposition 3.1.8.

$1 + A[[t]]^+$ is a λ-ring with the structure 3.1.7.

Proof.

Compare Atiyah-Tall [14] , p. 258.

Using this structure one sees that A is a special λ-ring if and only if λ_t is a λ-homomorphism. Moreover one has the Theorem of

Grothendieck that $1 + A[[t]]^+$ is a special λ-ring (Atiyah-Tall loc. cit.)

One can use 3.1.8 to show that certain λ-rings are special.

Proposition 3.1.9.

Let R be a λ-ring. Suppose that products of one-dimensional elements in R are again one-dimensional; in particular 1 shall be one-dimensional. Let $R_1 \subset R$ be the subring generated by one-dimensional elements. Then R_1 is a λ-subring which is special.

Proof.

Every element of R_1 has the form x-y where x,y are sums of one-dimensional elements, say $x = x_1 + \dots + x_p$, $y = y_1 + \dots + y_q$. Then $\lambda^i(x)$ is the i-th elementary symmetric function in the x_j hence a sum of one-dimensional elements. Moreover $\lambda^i(-y)$ is an integral polynomial in the $\lambda^j(y)$. Hence $\lambda^n(x-y) = \sum_i \lambda^i(x) \lambda^{n-i}(-y) \in R_1$. The remarks before 3.1.7 show that $\lambda_t \mid R_1$ is a ring-homomorphism and $\lambda_t \lambda^i(x) = \lambda^i \lambda_t(x)$ if x is a sum of one-dimensional elements and these two facts imply $\lambda_t \lambda^i(-x) = \lambda^i \lambda_t(-x)$ and then $\lambda_t \lambda^i(x-y) = \lambda^i \lambda_t(x-y)$.

Remark 3.1.10.

One can show (Atiyah-Tall [14]) - and later we shall use this fact - that a λ-ring R is special if and only if for any set $a_1, \dots, a_n$ of finite-dimensional elements in R there exists a λ-monomorphism $f : R \longrightarrow R'$ such that the fa_i are sums of one-dimensional elements. This is called the splitting principle for special λ-rings.

That a λ-ring structure, even if not special, may be very useful can be seen from the following Proposition due to G. Segal.

Proposition 3.1.11.

Let R be a λ-ring. Then all Z-torsion elements in R are nilpotent.

Proof.

Let a be a p-torsion element, say $p^n a = 0$. Then

$$1 = \lambda_t(o) = \lambda_t(a)^{p^n} = (1+at+\ldots)^{p^n} \equiv 1+a^{p^n}t^{p^n}+\ldots \mod p A$$

and hence $a^{p^n} = p\, b$ for some $b \in A$. Therefore

$$a^{(p^n+1)n} = (p\ a\ b)^n = (p^n a)(a^{n-1}b) = 0 .$$

3.2. Examples.

a) The integers may be given a λ-ring structure by defining $\lambda_t(1) = 1 + \sum m_n t^n$ where $m_1 = 1$. The canonical structure on Z is given by

$$\begin{aligned} \lambda_t(1) &= 1 + t \\ \lambda_t(m) &= (1+t)^m \\ \lambda^k(m) &= \binom{m}{k} \qquad m \geq 0 \\ \lambda^k(-m) &= (-1)^k \binom{m+k-1}{k} \end{aligned} \tag{3.2.1}$$

This canonical structure is special by 3.1.9. It can be given the following combinatorial interpretation: Let S be a set with m elements. Let $\Lambda^k S$ be the set of all subsets of cardianlity k. Then $|\Lambda^k S| = \binom{m}{k}$. The theory of special λ-rings may be thought of as an extremely elegant way of handling combinatorial identities for sets, symmetric functions, binomial coefficients, etc.

b) Let E,F be complex G-vector bundles over the (compact) G-space X where G is a compact Lie group. Then exterior powers Λ^i of G-vector bundles satisfy

$$\Lambda^{o}E = 1, \quad \Lambda^{1}E = E, \quad \Lambda^{n}(E \oplus F) = \bigoplus_{i=o}^{n} \Lambda^{i}(E) \otimes \Lambda^{j}(F) .$$

Let $K_G(X)$ be the Grothendieck ring of such G-vector bundles over X (Segal [142]). Then $E \longmapsto 1 + (\Lambda^1 E)t + (\Lambda^2 E)t^2 + \dots$ is a homomorphism from the additive semi-group of isomorphism classes of G-vector bundles over X into $1 + K_G(X)[[t]]^+$ and extends therefore uniquely to the Grothendieck group giving a map

$$\lambda_t : K_G(X) \longrightarrow 1+K_G(X)[[t]]^+ : x \longmapsto 1+ \lambda^1(x)t+\dots$$

such that $\lambda^i[E] = [\Lambda^i(E)]$ for E a G-vector bundle. These λ^i yield therefore a λ-ring structure on $K_G(X)$.

<u>Proposition 3.2.2.</u>

$K_G(X)$ <u>with this</u> λ-<u>structure is a special</u> λ-<u>ring</u>.

<u>Proof.</u>

The proof depends on the so called splitting principle which - especially for general G - is highly non-trivial. This splitting principle says: Given vector bundles $E_1,\dots,E_k$ over X. There exists a compact G-space Y and a G-map $f : Y \longrightarrow X$ such that the induced map $f^* : K_G(X) \longrightarrow K_G(Y)$ is injective and f^*E_i splits into a sum of line bundles. See Atiyah [9] , 2.7.11 or Karoubi [103], p. 193 for the case $G = \{1\}$.

Using the splitting principle 3.2.2 follows essentially from 3.1.9.

For a discussion of λ-operations in K-theory see also Atiyah [9] , ch. III, [7] ; Karoubi [103] IV. 7.

c) Other versions of topological K-theory like real K-Theory or

Real-K-Theory (Atiyah [8]), yield special λ-rings too.

d) A special case of b) is the representation ring R(G) of complex representations. Since representations are detected by restriction to cyclic subgroups and R(C) for a cyclic group C is generated by one-dimensional elements one can directly apply 3.1.9 to show that R(G) is special.

e) The Burnside ring acquires a λ-ring structure if we define $\lambda^i(S)$ for a finite G-set S to be the i-th symmetric power of S. We use the identity $\lambda^n(S+T) = \sum_i \lambda^i(S)\ \lambda^{n-i}(T)$ to extend this to A(G) as under b). This λ-ring structure is in general not special. See Siebeneicher [149] and the exercises to this section.

f) See Atiyah-Tall [14] , I. 2 for the construction of a free λ-ring on one generator.

3.3. γ-operations.

We assume that R is a special λ-ring. Then R contains a subring isomorphic to Z for if $1 \in R$ had finite additive order m, then $1 = \lambda_t(o) = \lambda_t(m\cdot 1) = (1+t)^m$ would give a contradiction (compare coefficients of t^m). A special λ-ring R is called <u>augmented</u> if there is given a λ-homomorphism $e : R \longrightarrow Z$. We call $I = \operatorname{Ker} e$ the <u>augmentation ideal</u>; it is a λ-ideal. Any element $x \in R$ may be written uniquely $x = e(x) + (x-e(x))$ with $e(x) \in Z$ and $x-e(x) \in I$.

Define the γ-<u>operations</u> on a special λ-ring R:

(3.3.1) $$\lambda_{t/(1-t)}(x) =: \gamma_t(x) = 1 + \sum_{n\geqslant 1} \gamma^i(x)t^i.$$

Then

(3.3.2) $$\gamma_t(x+y) = \gamma_t(x)\ \gamma_t(y).$$

Moreover one has

$$(3.3.3) \qquad \gamma^n(x) = \lambda^n(x+n-1).$$

Proof.

Using 3.2.1 we get

$$\lambda_{t/(1-t)}(x) = 1 + \sum_{i\geq 1} \lambda^i(x)\left(\sum_{k\geq o}\binom{i+k-1}{k}t^{k+i}\right)$$

$$= 1 + \sum_{j\geq 1}\left(\sum_{i=1}^{j} \lambda^i(x)\binom{j-1}{j-i}\right)t^j$$

$$= 1 + \sum_{j\geq 1} \lambda^j(x+j-1)t^j .$$

We conclude from 3.3.3 that $\lambda^j(x) = o$ for $j > n$ implies $\gamma^j(x-n)=o$ for $j > n$, i. e. if x is n-dimensional then x-n is of γ -dimension at most n.

Suppose R is an augmented λ -ring with augmentation $e : R \longrightarrow Z$ and augmentation ideal I = ker e. We define the γ-filtration by: $R_n \subset R$ is the additive group generated by monomials $\gamma^{n_1}(a_1)\cdot \ldots \cdot \gamma^{n_r}(a_r)$ where $a_i \in I$ and $\sum n_i \geq n$.

Proposition 3.3.4.

(i) $R_o = R$, $R_1 = I$.

(ii) $R_m R_n \subset R_{m+n}$.

(iii) R_n is a λ -ideal for $n \geq 1$.

Proof.

(i) and (ii) follow directly from the definitions. (iii): $R = Z \oplus R_1$ shows that R_n is an ideal. To show R_n is a λ -ideal, it is sufficient to show $\lambda^r(\gamma^m(x)) \in R_m$ for $x \in I$. First we compute for $i \geq m$

$$\lambda^i(x+m-1) = \gamma^i(x+m-i) = \sum_{s=o}^{i} \gamma^s(x)\ \gamma^{i-s}(m-i)$$

$$= \sum_{s=m}^{i} \gamma^s(x)\ \gamma^{i-s}(m-i) \in R_m$$

because $\gamma^{i-s}(m-i) = \lambda^{i-s}(m-s-1) = o$ for $i \geq m \geq s+1$. We use this in

$$\lambda^r(\gamma^m(x)) = \lambda^r(\lambda^m(x+m-1))$$

$$= P_{r,m}(\lambda^1(x+m-1), \dots, \lambda^{rm}(x+m-1))$$

and observe that $P_{r,m}(s_1,\dots,s_{rm})$ is a sum of monomials each containing a term s_i for $i \geq m$ because $P_{r,m}(s_1,\dots,s_{m-1},o,\dots,o) = o$.

Sometimes we want to work only with the augmentation ideal. We define: A ring I without identity is called a <u>special</u> γ -<u>ring</u> if there is an augmented special λ -ring R with I as augmentation ideal. I then carries the induced γ^i-operations. We define the γ-filtration as before, I_n being the ideal generated by monomials $\gamma^{n_1}(a_1)\cdot\ldots\cdot\gamma^{n_r}(a_r)$ where $a_i \in I$, $\sum n_i \geq n$. We have

(3.3.5) $$I_1 = I,\ I_m I_n \subset I_{m+n},\quad \gamma^i(I_n) \subset I_n.$$

<u>3.4. The Adams operations.</u>

Adams introduced in [1] certain operations derived from the λ^i which are much easier to handle algebraically.

Let R be a special λ -ring. Define maps

$$\psi^n : R \longrightarrow R, \qquad n \geq 1$$

by

$$(3.4.1) \qquad \psi_{-t}(x) = -t \frac{d}{dt}(\lambda_t(x))/\lambda_t(x)$$

$$\psi_t(x) = \sum_{n\geq 1} \psi^n(x)t^n.$$

A more elementary way of defining the ψ^n is: Define the <u>Newton polynomial</u>

$$N_n(s_1,\ldots,s_n) = \sum_{j=1}^n x_j^n$$

where s_i is the i-th elementary symmetric function of the x_j. Then put

$$(3.4.2) \qquad \psi^n(x) = N_n(\lambda^1(x),\ldots,\lambda^n(x)).$$

We leave it as an exercise to show that the two definitions are equivalent.

We want to show that the ψ^n are λ-ring homomorphisms. This means we have to verify certain identities between the ψ^n- and λ^j-operations. We use the <u>verification principle</u> which says that it is enough to verify the identities on elements which are sums of one-dimensional elements. A formal proof of this principle is given in Atiyah-Tall [14], I. 3.4, I. 4.5. Since in the applications the λ-rings are finite-dimensional and since we have to prove the splitting principle in order to show that something is a special λ-ring we do not prove the verification principle.

<u>Proposition 3.4.3.</u>

(i) <u>If</u> x <u>is one-dimensional then</u> $\psi^n x = x^n$.

(ii) ψ^n <u>is a</u> λ-<u>homomorphism</u>.

(iii) $\psi^m\psi^n = \psi^n\psi^m = \psi^{mn}$.

(iv) $\psi^{p^r}(x) \equiv x^{p^r} \bmod p$ (p prime).

Proof.

(i) follows directly from 3.4.2.

(ii) Suppose x_i, y_j are one-dimensional. Then $x_i y_j$ is one-dimensional because R is special. From 3.4.1 one obtains that ψ^n is an additive homomorphism. Moreover

$$\psi^n(\Sigma x_i \Sigma y_j) = \psi^n(\Sigma x_i y_j) = \Sigma \psi^n(x_i y_j) = \Sigma (x_i y_j)^n$$
$$= (\Sigma x_i^n)(\Sigma y_j^n) = \psi^n(\Sigma x_i)\,\psi^n(\Sigma y_j).$$

$$\psi^n(\lambda^m(\Sigma x_i)) = \psi^n(s_m(x_1,\dots,x_r)) = s_m(x_1^n,\dots,x_r^n)$$
$$= \lambda^m(\Sigma x_i^n) = \lambda^m(\psi^n(\Sigma x_i)).$$

Now use the verification principle.

(iii) and (iv) are likewise immediate from the verification principle.

As a consequence we have ψ^n on a special γ-ring. Moreover the ψ^n preserve the γ-filtration.

Proposition 3.4.4.

Let I be a special γ-ring. Assume $x \in I_n$. Then the following holds:

(i) $\psi^k(x) - k^n x \in I_{n+1}$

(ii) $\psi^k(x) + (-1)^k k\, \lambda^k(x) \in I_{n+1}$

(iii) $\lambda^k(x) + (-1)^k k^{n-1} x \in I_{n+1}$.

Proof.

(i) We need only show that $\psi^k(\gamma^m(a)) - k^m \gamma^m(a) \in I_{m+1}$ for $a \in I$, because ψ^k is a γ-homomorphism. If $x_1,\dots,x_r$ have γ-dimension one, i. e. $\gamma_t(x_i) = 1+x_i t$, then $1+x_i$ has λ-dimension one, hence $\psi^k(x_i) = (1+x_i)^k - 1$ and therefore

$$\psi^k(\gamma^m(x_1+\dots+x_r)) - k^m \gamma^m(x_1+\dots+x_r)$$

$$= \psi^k(s_m(x_1,\dots,x_r)) - k^m s_m(x_1,\dots,x_r)$$

$$s_m((1+x_1)^k - 1,\dots,(1+x_r)^k - 1) - k^m s_m(x_1,\dots,x_r).$$

This is a symmetric polynomial of degree $\geq m+1$, hence (i) is true for $x = \sum x_i$ and, by the verification principle, therefore in general.

(ii) From the Newton polynomials we obtain the well-known identity

$$\psi^k(x) - \psi^{k-1}(x)\,\lambda^1(x)+\dots+(-1)^{k-1}\psi^1(x)\,\lambda^{k-1}(x)+(-1)^k k\,\lambda^k(x)=0$$

which implies the result, because $\psi^i(x) \in I_n$, $\lambda^i(x) \in I_n$ for $i \geq 1$, and $x \in I_n$.

(iii) From (i) and (ii) we obtain $k\lambda^k(x) + (-1)^k k^n(x) \in I_{n+1}$. Thus the result follows if there is no k-torsion. (One can produce suitable universal situations without torsion, e. g. free λ-rings; thus one gets the result in general. One should note that the assertions are natural with respect to λ-homomorphisms.)

3.5. Adams-operations on representation rings.

Let G be a finite group and R(G;F) be the Grothendieck ring (= representation ring) of finitely generated F[G]-modules where F is a field. We assume for simplicity that F has characteristic zero. Then elements in R(G;F) are determined by their character. We identify R(G;F) with the corresponding character ring. Exterior powers define a special λ-ring structure on R(G;F). We want to compute the associated Adams-operations.

Proposition 3.5.1.

Let $x \in R(G;F)$. Then

$$\psi^k x(g) = x(g^k), \qquad g \in G.$$

In particular

$$\psi^k = \psi^{k+|G|}$$

Proof.

Restrict to the cyclic group C generated by g. Pass to an algebraic closure of F so that $x \mid C = y-z$ where y and z are sums of one-dimensional representations. The result then follows from 3.4.3 taking into account that for a one-dimensional representation x the relation $x^k(g) = x(g^k)$ holds.

Now assume that $F = Q[\zeta_n]$ where ζ_n is a primitive n-th root of unity. Assume that k is prime to the group order $|G|$. The Galois group $Gal(Q[\zeta] : Q)$ is isomorphic to Z/nZ^*, namely so that k mod n corresponds to the field automorphism P^k characterized by $P^k(\zeta_n) = \zeta_n^k$. Since characters of $F[G]$-modules take values in $Q[\zeta_n]$ we can apply P^k to such characters. Let $Q[\zeta_n]$ be a splitting field for G. (By a famous theorem of Brauer it suffices to take for n the exponent of G; see Serre [147], p. 109). Then we show

Proposition 3.5.2.

(i) $\psi^k x = P^k x$ for $x \in R(G;Q[\zeta_n])$ and $(k,|G|) = 1$.

(ii) If x is the character of an irreducibel module then $\psi^k x$ is irreducible too (again k prime to $|G|$).

Proof.

(i) Let x be the character of a matrix representation. Restrict to the cyclic subgroup C generated by $g \in G$. Then the matrix for g is equivalent to a diagonal matrix with roots of unity $u_1, \ldots, u_r$ on the diagonal. Then $\psi^k(x)(g) = \sum u_i^k = P^k(\sum u_i) = P^k(x(g))$.

(ii) Apply the Galois automorphism P^k to a matrix representation over $Q[\zeta_n]$.

Remark 3.5.3.

The Adams operation are, of course, independent of the field of definition. Therefore 3.5.2 holds more generally.

3.7. The Bott cannibalistic class θ_k.

Let R be a special λ-ring and let ζ_k be a primitive k-th root of unity. Let $P(R) \subset R$ be the subset of finite-dimensional elements in R. Then P(R) is an additive semi-group. If $x \in P(R)$ we consider the product

(3.7.1) $\quad \theta_k(x) := \prod_u \lambda_{-u}(x) \in R \otimes_Z Z[\zeta_k]$

where the product is taken over all roots of $t^k - 1 = 0$ except 1. We identify R with its image in $R \otimes Z[\zeta_k]$ under the canonical map $r \longmapsto r \otimes 1$. Then $\theta_k(u)$ is contained in R. [In order to see this consider the following diagram

$$\begin{array}{ccc} R \otimes_Z Z[s_1, \ldots, s_{k-1}] & \longrightarrow & R \otimes_Z Z[t_1, \ldots, t_{k-1}] \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \otimes_Z Z[\zeta_k] \end{array}$$

where $t_1, \ldots, t_{k-1}$ are indeterminates and $s_1, \ldots, s_{k-1}$ are the elementary symmetric functions in the t_j. The vertical maps are induced by substituting for $t_1, \ldots, t_{k-1}$ the roots of $t^k - 1 = 0$ except 1. Then

$\Pi_j \lambda_{-t_j}(x)$ is symmetric in the t_j and since $Z[s_1,\dots,s_{k-1}] \subset Z[t_1,\dots,t_{k-1}]$ is an inclusion as a direct summand we see that $\Pi_j \lambda_{-t_j}(x) \in R \otimes_Z Z[s_1,\dots,s_{k-1}]$. But the map at the bottom is an injection too because $Z \longrightarrow Z[\zeta_k] : n \longmapsto n$ is a direct injection.] We call it the Bott cannibalistic class θ_k. The following is immediate from the definition.

Proposition 3.7.2.

(i) If x is one-dimensional then

$$\theta_k(x) = 1 + x + \dots + x^{k-1}.$$

(ii) If $x,y \in P(R)$ then

$$\theta_k(x+y) = \theta_k(x)\ \theta_k(y) .$$

Since $\theta_k(1) = k$ θ_k is not in general a unit in R so that θ_k cannot be extended to the additive subgroup generated by finite-dimensional elements. In the next section on p-adic γ-rings we find a remedy for this defect.

3.8. p-adic γ-rings.

Let p be a prime number. Let Z_p denote the p-adic integers. One can define Z_p as the inverse limit ring inv lim Z/p^nZ. If A is a finitely generated abelian group then $A \otimes_Z Z_p$ is cannonically isomorphic to the p-adic completion of A

$$A_p := \text{inv lim } A/p^n A.$$

Tensoring with Z_p is an exact functor on the category of finitely generated abelian groups. (See Atiyah-Mac Donald [11] , Ch. 10 for this and other back ground material on completions.) Groups $A_p^\wedge$ carry the p-adic topology: a fundamental system of neighbourhoods of zero is given by the subgroups $p^n A_p^\wedge$. They are complete and Hausdorff in this topology.

If B is a special γ-ring, then, by definition, there is a special augmented λ-ring R such that B = ker e where e is the augmentation. Then we have the exact sequence (because e : R $\longrightarrow$ Z splits)

$$0 \longrightarrow B \otimes Z_p \longrightarrow R \otimes Z_p \longrightarrow Z_p \longrightarrow 0.$$

We want to define the structure of a special λ-ring on $R \otimes Z_p$ such that $B \otimes Z_p$ is a λ-ideal. We can extend the λ^i by continuity if we have shown

Proposition 3.8.1.

The λ^i are continuous with respect to the p-adic topology.

Proof.

Given i and N chose k_o such that $\binom{p^k}{j}$ is divisible by p^N for $k \geqslant k_o$ and $1 \leqslant j \leqslant i$. Then

$$\lambda^j(p^k x) = P_j(\lambda^1(p^k), \ldots, \lambda^j(p^k); \lambda^1(x), \ldots, \lambda^j(x))$$

is contained in p^N R if $k \geqslant k_o$ and $1 \leqslant j \leqslant i$ because P_j is of weight j in the first j variables. If $x-y = p^k z$ then

$$\lambda^i(y) - \lambda^i(x) = \sum_{j=1}^{i} \lambda^{i-j}(y)\, \lambda^j(p^k z) \in p^N R$$

for $k \geqslant k_o$.

The proof of this Proposition shows that if $a \in Z_p$ is the limit of a sequence (a_n), $a_n \in Z$ then $\lim \lambda^i(a_n x) = \lambda^i(\lim a_n x) = \lambda^i(ax)$ and hence

$$\lambda_t(ax) = \lambda_t(x)^a \qquad a \in Z_p$$

(3.8.2)
$$\gamma_t(ax) = \gamma_t(x)^a \qquad x \in R$$

$$\psi^k(ax) = a\,\psi^k(x).$$

After these preliminary remarks we define a p-*adic* γ-*ring* A to be a γ-ring which is the completion $A = B \otimes Z_p$ of some γ-ring B which is finitely generated as an abelian group; moreover we require that *the* γ-*topology on* B *is finer than the* p-*adic topology*.

We now describe some examples of p-adic γ-rings.

Proposition 3.8.3.

Let X *be a finite connected* CW-*complex. Then the* n-th γ-*filtration on* $\tilde{K}(X)$ *is contained in the* n-th *skeleton-filtration. In particular the* γ-*topology is discrete and* $\tilde{K}(X) \otimes Z_p$ *is a* p-*adic* γ-*ring*.

Proof.

Let X^n be the n-skeleton on X. Then the n-th skeleton filtration $S_n\tilde{K}(X)$ is defined to be the kernel of the restriction map $i^* : \tilde{K}(X) \longrightarrow \tilde{K}(X^{n-1})$. Any element of $K(X^{n-1})$ is represented by an element $x = [E] - (n-1)$ where E is an (n-1)-dimensional bundle. Hence $i^* \gamma^n(y) = \gamma^n(i^* y) = \gamma^n(E-n+1) = 0$. The relation $S_n S_m \subset S_{n+m}$ then implies the result.

Let R(G) be the representation ring of the finite group G over the complex numbers. Let $R(G) \longrightarrow Z : x \longmapsto \dim x$ be the augmentation with kernel I(G). Then we can consider three topologies on R(G):

(i) The p-adic topology.

(ii) The I(G)-adic topology.

(iii) The γ-topology, defined by the γ-filtration.

Proposition 3.8.4.

Let G be a p-group. Then the topologies (i), (ii), and (iii) coincide. In particular $I(G) \otimes Z_p$ is a p-adic γ-ring.

We use the next Proposition for the proof of 3.8.4.

Proposition 3.8.5.

Let I be a γ-ring which is generated by a finite number of elements with finite γ-dimension. Then the I-adic topology coincides with the γ-topology.

Proof.

By definition of the γ-filtration we have $I_n \subset I^n$. Let m be the maximal γ-dimension of a given finite set of generators for I. Then γ^{m+1} applied to the monomials in the generators must lie in I^2. Since $\gamma^{m+1}(-x) \equiv -\gamma^{m+1}(x) \mod I^2$ we obtain $I_{m+1} \subset I^2$. By induction one shows $I_{km+1} \subset I^k$.

Proof of 3.8.4.

Put I = I(G). By 3.8.5 the topologies (ii) and (iii) coincide. Let $m = |G|$. Then

$$(x-e(x))^m \equiv x^m-e(x)^m \mod p\ R(G)$$

because m is a p-power. By 3.5.1 we have $\psi^m x = e(x)$ and by 3.4.3 (iv) we have $\psi^m x \equiv x^m \mod p\ R(G)$. Putting these facts together we obtain

$$(x-e(x))^m \equiv e(x) - e(x)^m \equiv 0 \mod p\ R(G) .$$

This shows $I^m \subset p\,I$, hence the I-adic topology (and therefore the γ-topology) is finer than the p-adic topology. One can show that $mI \subset I^2$ (see Atiyah [6]), so that the p-adic topology is also finer than the I-adic. (This last fact also follows from localization theorems to be proved later in this lecture.)

As a slight generalization of 3.8.4 we mention

<u>Proposition 3.8.6.</u>

<u>Let</u> G <u>be a</u> p-<u>group and</u> X <u>a connected finite</u> G-CW-<u>complex</u>. <u>Then</u> $\tilde{K}_G(X) \otimes Z_p$ <u>is a</u> p-<u>adic</u> γ-<u>ring</u>. ($\tilde{K}_G(X)$ = <u>kernel of</u> $x \longmapsto \dim x$)

<u>Proof</u> (sketch).

From the fact that X is a finite G-CW-complex one shows by induction over the number of cells that $K_G(X)$ is a finitely generated abelian group. By 3.8.5 the γ-topology coincides with the $\tilde{K}_G(X)$-adic topology. Let X^o be the equivariant zero-skeleton of X. The kernel N of $r : K_G(X) \longrightarrow K_G(X^o)$ is nilpotent (compare Segal [142] , Proposition 5.1). Moreover $K_G(X^o) \cong \prod R(G_x)$, the product taken over the orbits of X^o. Put $I = \tilde{K}_G(X)$. By Atiyah-Mac Donald [11] , Theorem 10.11, the p-adic topology on rI is induced from the p-adic topology on $K_G(X^o)$. Hence from 3.8.4 we see that for some t, $rI^t \subset prI$, or equivalently, $I^t \subset pI + N$. But if $N^k = 0$ then $I^{tk} \subset (pI+N)^k \subset pI$. This shows that the I-adic topology is finer than the p-adic topology.

Now we continue with the general discussion of p-adic γ-rings $A = B \otimes Z_p$. If B_n is the n-th γ-ideal of B we let $A(n) = B_n \otimes Z_p$ be its closure. From 3.8.1 we obtain that the A(n) are γ-ideals. By definition of a p-adic γ-ring the topology defined by the system $A(n)$, $n \geq 1$, is finer than the p-adic topology; in particular this topology is also Hausdorff and one has

$$A \cong \text{inv lim } A/A(n). \tag{3.8.7}$$

$A(n)$ contains the n-th γ-ideal A_n of A but A_n need not be closed in the p-adic topology. We observe

$$A(n)/A(n+1) \cong (B_n/B_{n+1}) \otimes Z_p \tag{3.8.8}$$

because $\otimes Z_p$ is exact on finitely generated abelian groups. From 3.4.4 and 3.8.8 we obtain

Proposition 3.8.9.

$A(n)/A(n+1)$ is a p-adic γ-ring. The product of two elements is zero. For $a \in A(n)/A(n+1)$ we have

$$\lambda^k(a) = (-1)^{k-1} k^{n-1} a$$

$$\psi^k(a) = k^n a.$$

We shall show that γ^k acts on $A(n)/A(n+1)$ as multiplication with a certain constant $c(k,n)$ independent of the ring A. From $\gamma^k(x) = \lambda^k(x+k-1)$ one computes

$$c(k,n) = \sum_{i=1}^{k} (-1)^{i-1} i^{n-1} \binom{k-1}{k-i} . \tag{3.8.10}$$

In order to analyse these numbers we put

$$\gamma_t(x) = 1 + f_n(t)x$$

where

$$f_n(t) = \sum_{j=1}^{\infty} (-1)^{j-1} j^{n-1} \left(\frac{t}{1-t}\right)^j$$

is a certain formal power series in $Z[[t]]$. For n = 1 this is a geometric series with sum

$$f_1(t) = t .$$

If we differentiate $f_n(t)$ formally with respect to t we obtain the recursion formula

$$f_{n+1}(t) = t(1-t)\, f_n'(t)$$

so that $f_n(t)$ is actually a polynomial of degree n

$$f_n(t) = \sum_{j=1}^{n} c(j,n)t^j .$$

In particular $\gamma^m = o$ on $A(n)/A(n+1)$ for $m > n$.

3.9. The operation ρ_k.

We describe a variant of the Bott map θ_k for p-adic γ-rings A. A topology shall always be the p-adic topology if not otherwise specified.

A series $\sum_{r \geq 1} a_r$, with $a_r \in A(r)$, converges in the p-adic topology since it converges in the filtration topology $(A(n) \mid n \geq 1)$ which is finer. Therefore the set 1 + A of symbols 1 + a, $a \in A$, with multiplication $(1+a)(1+b) = 1+a+b+ab$ is a group. It is a compact, topological group, with neighbourhood basis of 1 given by $(1+p^nA \mid n \geq 0)$, or equivalently $(1+p^nA+A(n) \mid n \geq 1)$.

Let k be a natural number prime to p. Consider $Z_p[\zeta_k]$ where ζ_k is a primitive k-th root of unity in an algebraic closure of the p-adic numbers. The product $\prod(1-u)$ over all roots u of $t^k - 1 = 0$ except 1 is equal to k, hence a unit in Z_p. Therefore 1-u is a unit in $Z_p[\zeta_k]$ and hence $u/(u-1) \in Z_p[\zeta_k]$. The series

$$\gamma_{u/(u-1)}(a) = 1 + \gamma^1(a)\, u/(u-1) + \gamma^2(a)(u/u-1)^2 + \dots$$

converges in the p-adic topology on $1 + A \otimes_{Z_p} Z_p[\zeta_k]$ hence defines an element $\gamma_{u/(u-1)}(a)$ in this multiplicative group. We define

(3.9.1) $$\rho_k(a) = \prod \gamma_{u/(u-1)}(a) \in 1 + A \otimes Z_p[\zeta_k]$$

where the product is taken over all roots of $t^k - 1 = 0$ except 1. The Z_p-algebra $Z_p[\zeta_k]$ is free as Z_p-module with $Z_p \cdot 1$ as a direct summand; therefore $A = A \otimes_{Z_p} Z_p \subset A \otimes_{Z_p} Z_p[\zeta_k]$ as a subring. (As to the freeness of the module: Let $L \in Q_p[t]$ be an irreducible polynomial with $L(\zeta_k) = 0$. Then L divides the cyclotomic polynomial ϕ_k. Since Z_p is factorial we can choose for L a monic polynomial in $Z_p[t]$, by the Gauß-Lemma. Then $Z_p[\zeta_k] \cong Z_p[t]/L$ and the right-hand side is clearly a free module.) We claim: $\rho_k(a) \in 1 + A$. This follows from the fact that a coefficient of a monomial in the $\gamma^i(a)$ in the expansion of $\rho_k(a)$ according to definition 3.9.1 is symmetric in the roots of $t^k - 1 = 0$ (compare 3.7).

<u>Proposition 3.9.2.</u>

The map

$$\rho_k : A \longrightarrow 1 + A$$

from the additive compact group A into the multiplicative compact group 1 + A is a continuous homomorphism. It commutes with the Adams operations and maps A(n) into 1 + A(n).

Proof.

ς_k is a homomorphism: directly from 3.3.2 and 3.9.1. Since $\varsigma_k(p^n a) = (\varsigma_k(a))^{p^n}$ and $(1+a)^{p^n} \in 1 + p^N + A(N)$ if $\binom{p^n}{i} \equiv 0 \bmod p^N$ for $1 \leq i \leq N$ we see that ς_k is p-adically continuous. Since ψ^j commutes with the γ^i it commutes with ς_k. Since A(n) is a γ-ideal $\varsigma_k A(n) \subset 1 + A(n)$.

Remark 3.9.3.

If A is a ring without identity we can adjoin an identity in the standard manner: On the additive group Z × A define a multiplication $(m,a)(n,b) = (mn, mb+na+ab)$. Then $1 + A = \{(1,a) \mid a \in A\} \subset Z \times A$. If $B \subset A$ is an ideal and if 1 + B and 1 + A are groups then $(1+A)/(1+B) \cong 1 + A/B$.

3.10. Oriented γ-rings.

A γ-ring A is said to be oriented if

$$\gamma_t(a) = \gamma_{1-t}(a), \qquad a \in A. \tag{3.10.1}$$

This terminology has the following reason: Suppose A is the augmentation ideal of the special augmented finite-dimensional λ-ring R. Then

Proposition 3.10.2.

A is oriented if and only if for every finite-dimensional element x, of dimension n say, $\lambda^r(x) = \lambda^{n-r}(x)$ for all r.

Proof.

If 3.10.1 is satisfied for a_1 and a_2 then for $a_1 - a_2$ too. The equation $\lambda^r(x) = \lambda^{n-r}(x)$ implies $\lambda_t(x) = t^n \lambda_{1/t}(x)$ and this yields

$$\gamma_t(x-n) = \lambda_{t/(1-t)}(x-n) = \lambda_{t/(1-t)}(x)(1-t)^n$$

$$= t^n \lambda_{(1-t)/t}(x)$$

$$\gamma_{1-t}(x-n) = \lambda_{(1-t)/t}(x-n) = \lambda_{(1-t)/t}(x)(1+(1-t)/t)^{-n}$$

$$= t^n \lambda_{(1-t)/t}(x) .$$

Note that n must be the augmentation of an n-dimensional element x because $\lambda^n(x) = 1$, so that $x-n \in A$. The same calculation gives $\lambda^r(x) = \lambda^{n-r}(x)$ from 3.10.1.

We call R an <u>oriented</u> λ-<u>ring</u> if $\lambda^r(x) = \lambda^{n-r}(x)$ whenever x is n-dimensional.

<u>Example 3.10.3.</u>

Let $KO_G(X)$ be the Grothendieck ring of real G-vector bundles over the compact G-space X where G is a compact Lie group. An n-dimensional G-vector bundle E is called orientable if the n-th exterior power $\Lambda^n E$ is the G-vector bundle $X \times \mathbb{R} \longrightarrow X$ with trivial G-action on $\mathbb{R}$. If E is orientable then $\Lambda^r E \cong \Lambda^{n-r} E$. Hence

$$KSO_G(X) = \{ E - F \in KO_G(X) \mid E,F \text{ orientable} \}$$

is an oriented λ-ring and the associated augmentation ideal is an oriented γ-ring.

If x is a one-dimensional element in the oriented λ-ring then $\lambda^1(x) = \lambda^0(x) = 1$. Therefore one should think of such a ring as containing essentially only even-dimensional elements.

We now consider a refinement of the operations Θ_k(resp. ς_k) for an oriented λ-ring R (a p-adic oriented γ-ring A).

Let $x \in R$ be an element of dimension 2m. Let k be an odd integer. Let J a set of k-th roots of unity $u \neq 1$ which contains from each pair u, u^{-1} exactly one element. (Since $k \equiv 1(2)$ we have $u \neq u^{-1}$.) The product $k^m \prod_{u \in J}(1-u)^{-2m}$ is an algebraic integer because $\prod_{u \neq 1}(1-u) = k$. Therefore

$$(3.10.4) \qquad k^m \prod_{u \in J} \lambda_{-u}(x)(1-u)^{2m} \in R[\zeta_k]$$

where ζ_k is a primitive k-th root of unity. The fact that R is oriented implies

$$(3.10.5) \qquad \lambda_{-u}(x)(1-u)^{-2m} = \lambda_{-1/u}(x)(1-1/u)^{-2m}.$$

Therefore 3.10.4 is independent of the choice of J. We call this element

$$\Theta_k^{or}(x) .$$

<u>Proposition 3.10.6.</u>

(i) *If x and y are even-dimensional then* $\Theta_k^{or}(x+y) = \Theta_k^{or}(x)\,\Theta_k^{or}(y)$.

(ii) *The square of* $\Theta_k^{or}(x)$ *is* $\Theta_k(x)$.

(iii) $\Theta_k^{or}(x) \in R$.

<u>Proof.</u>

(i) follows directly from the analogous property of λ_t. (ii) follows from the definitions, using 3.10.5. (iii) Using 3.10.5 again one can see that $\Theta_k^{or}(x)$ is formally invariant under the Galois group of $Q(\zeta_k)$ over Q.

If A is an oriented p-adic γ-ring one defines the square root of ρ_k by

$$(3.10.7) \qquad \rho_k^{or}(x) = \prod_{u \in J} \gamma_{u/u-1}(x) .$$

Using $\gamma_t = \gamma_{1-t}$ one shows that the following holds

<u>Proposition 3.10.8.</u>

(i) $\rho_k^{or}(x+y) = \rho_k^{or}(x) \; \rho_k^{or}(y)$.

(ii) <u>The square of</u> $\rho_k^{or}(x)$ <u>is</u> $\rho_k(x)$.

(iii) $\rho_k^{or}(x) \in 1 + A$.

We now compute $\theta_k^{or}(z)$ for a two-dimensional element z. We have $\lambda_{-u}(z) = 1 - uz + u^2$. If we formally write $z = x+y$ with $xy = 1$ then $\lambda_{-u}(z) = (1-ux)(1-uy)$ and therefore

$$(3.10.9) \qquad \lambda_{-u}(z)(1-u)^{-2} = y \, \frac{1-ux}{1-u} \cdot \frac{1-u^{-1}x}{1-u^{-1}} .$$

If we multiply these expressions according to the definition of $\theta_k^{or}(z)$ we obtain

$$(3.10.10) \qquad \theta_k^{or}(z) = ky^{(k-1)/2} \prod_u (1-ux) \prod_u (1-u)^{-1}$$

$$= y^{(k-1)/2}(1+x+\ldots+x^{k-1})$$

$$= (x^{(k-1)/2} + x^{(k-3)/2} + \ldots + y^{(k-1)/2}).$$

This last expression may also be written

$$(3.10.11) \qquad \frac{x^{k/2} - x^{-k/2}}{x^{1/2} - x^{-1/2}}$$

where we use this at this point merely as a suggestive formula without having $x^{1/2}$ defined. Actually $\Theta_k^{or}(z)$ is an integral polynomial in z: The polynomial

$$P_k(t) = \prod_{u \in J} (t-(u+u^{-1}))$$

is contained in $Z[t]$ and has degree $(k-1)/2$, e. g. $P_3(t) = 1+t$, $P_5(t) = -1+1+t^2$. One has for a 2-dimensional z

(3.10.12) $$\Theta_k^{or}(z) = P_k(z).$$

A proof follows from the identity

$$t^{k-1} P_k(t^2+t^{-2}) = (1+t+...+t^{2k-1})/(1+t)$$

which can be seen by observing that both sides are monic polynomials of degree 2k-2 having the 2k-th roots of unity $= \pm 1$ as roots.

From 3.10.10 one obtains for a 2-dimensional z the identiy

(3.10.13) $$\Theta_k^{or}(z) = 1 + \psi^1 z + \psi^2 z + ... + \psi^{(k-1)/2} z .$$

3.11. The action of ρ_k on scalar γ-rings.

We consider p-adic γ-rings A with trivial multiplication, like A(n)/A(n+1) in Proposition 3.8.9, on which ψ^k is multiplication by k^n and λ^k multiplication by $(-1)^{k-1}k^{n-1}$. Then we have seen in 3.8. that

$$\gamma_t(x) = 1 + f_n(t)x$$

where $f_n(t)$ in an integral polynomial defined by the recursion formula

$$f_1(t) = t, \qquad f_{n+1}(t) = t(1-t)f_n'(t).$$

Therefore ρ_k is given by

$$\rho_k(x) = \prod_u (1+xf_n(\tfrac{u}{u-1})) = 1 + x \sum_u f_n(\tfrac{u}{u-1})$$

We have to compute the rational number (Galois theory)

$$\sum_u f_n(\tfrac{u}{u-1}) =: b_n(k) ,$$

the sum being taken over the k-th roots of unity $u \neq 1$. Put $h_n(t) =$ $= f_n(\frac{t}{t-1})$.

<u>Proposition 3.11.1.</u>

<u>We have the following identity between formal power series in</u> x <u>and</u> t <u>over</u> Q

$$\log(1 + \frac{t}{1-t}(1-e^x)) = \sum_{n \geq 1} h_n(t) \frac{x^n}{n!} .$$

(The meaning of the left hand side is: Use the power series $\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots$ and replace y with the power series $\frac{t}{1-t}(1-e^x)$ which has no constant term.)

<u>Proof.</u>

We put

$$K(t,x) := \log(1+ \frac{t}{1-t}(1-e^x)) = \sum_{n \geq 1} g_n(t) \frac{x^n}{n!}$$

where the $g_n(t)$ are certain power series in t. We differentiate $K(t,x)$ with respect to t and x and obtain

$$\frac{dK}{dt} = \frac{e^x}{te^x-1} + \frac{1}{1-t} \, , \quad \frac{dK}{dx} = \frac{te^x}{te^x-1}$$

hence

$$t\,\frac{dK}{dt} - \frac{dK}{dx} = \frac{t}{1-t} \; .$$

We apply this differential equation to $\sum_{n\geq 1} g_n(t)\,\frac{x^n}{n!}$ and compare coefficients, thus obtaining

$$g_1(t) = -\frac{t}{1-t}$$

$$g_n(t) = tg'_{n-1}(t)$$

and these are precisely the recursion formulas for the h_n.

If we replace t in 3.11.1 with a k-th root of unity $u \neq 1$ we obtain an identity between formal power series in x over $Q(\zeta_k)$. We compute the $b_n(k)$ as follows

$$\sum_{n\geq 1} b_n(k)\,\frac{x^n}{n!} = \sum_{u\neq 1} \log \frac{1-ue^x}{1-u}$$

$$= \log \prod_{u-1} \frac{1-ue^x}{1-u} = \log \frac{1}{k}\,(1+e^x+\ldots+e^{(k-1)x})$$

$$= \log \frac{e^{kx}-1}{kx} - \log \frac{e^x-1}{x}$$

$$= \sum_{n\geq 1} (k^n-1)\; a_n\,\frac{x^n}{n!}$$

if we use the expansion $\log \frac{e^x-1}{x} = \sum_{n\geq 1} a_n\,\frac{x^n}{n!}$.

The a_n are easily expressed in terms of <u>Bernoulli numbers</u> B_m which are defined by

$$\frac{t}{e^t-1} = 1 + \sum_{m \geq 1} B_m \frac{t^m}{m!} .$$

This yields immediately $B_1 = -\frac{1}{2}$, $B_{2m+1} = 0$ for $m \geq 1$. If we differentiate the defining series of the a_n with respect to x we obtain

$$\sum_{n \geq 1} na_n \frac{x^{n-1}}{n!} = 1 - \frac{1}{x} + \sum_{n \geq 0} B_n \frac{x^{n-1}}{n!}$$

and then

$$a_n = \frac{B^n}{n} \quad \text{for} \quad n > 1,\ a_1 = \frac{1}{2} .$$

Collecting these computations we obtain

<u>Proposition 3.11.2.</u>

$\varsigma_k : A(n)/A(n+1) \longrightarrow 1 + A(n)/A(n+1)$ <u>is the map</u>

$$x \longmapsto 1 + (k^n-1) \frac{B_n}{n} x .$$

We now come to oriented γ-rings. From the recursion formula for the rational functions $h_n(t)$ one proves by induction

(3.11.3) $$h_m(t^{-1}) = (-1)^m h_m(t)$$

$$f_m(t) = (-1)^m f_m(t) .$$

The previous calculations yield

Proposition 3.11.4.

Let A be an oriented p-adic γ-ring. Then
$\rho_k^{or} : A(2n)/A(2n+1) \longrightarrow 1 + A(2n)/A(2n+1)$ *is the map*

$$x \longmapsto 1 + (k^{2n}-1)\,\frac{B_{2n}}{4n}\,x\ .$$

Remark 3.11.5.

Equating coefficients in $\sum \gamma^r(a)t^r = \sum \gamma^r(a)\ (1-t)^r$ one finds

$$\gamma^k = (-1)^k\,\gamma^k + (-1)^k(k+1)\,\gamma^{k+1} + c$$

where c has γ-filtration at least k+2. This gives by induction
$A(2n-1) = A(2n)$ for $n \geq 1$.

3.12. The connection between θ_k and ρ_k.

The map θ_k was only defined for finite-dimensional elements x. In order to extend it to negatives of such elements one must have that $\theta_k(x)$ is a unit. This can sometimes be accomplished by passing to the p-adic completion. We describe the formal setting.

Let R be an augmented special λ-ring with augmantation $e : R \longrightarrow Z$ and augmentation ideal B = ker e. Moreover we assume:

(i) R is finitely generated as an abelain group by $x_1 = 1, x_2,\ldots,x_m$ which are finite-dimensional.

(ii) $e(x_r) = \dim x_r$ for $r = 1,\ldots,m$.

(iii) The γ-topology on B is finer than the p-adic topology.

We then have $e(x) = \dim x$ whenever x is finite-dimensional and moreover $\gamma_t(x-e(x))$ is a polynomial in t of degree $\leq \dim x$, hence

$$\gamma\text{-}\dim(x-e(x)) \leq \dim x.$$

Proposition 3.8.5 shows that the B-adic topology coincides with the γ-topology. The ring $A = B \otimes Z_p$ is a p-adic γ-ring, by (iii) above.

<u>Proposition 3.12.1.</u>

<u>Let</u> $i : R \longrightarrow R \otimes Z_p$ <u>be the canonical map and</u> $(k,p) = 1$. <u>Then for finite-dimensional</u> $x \in R$ <u>the element</u> $i\,\theta_k(x)$ <u>is a unit in</u> $R \otimes Z_p$.

<u>Proof.</u>

If dim $x = n$ then $e\theta_k x = \theta_k ex = \theta_k n = k^n$. Put $r = k^n$, then $(r,p) = 1$ and r^{-1} exists in Z_p. Therefore $r^{-1} i\theta_k x = 1+a$, $a \in B \otimes Z_p$. But $1+A \subset B \otimes Z_p$ is a multiplicative subgroup. If $(1+a)(1+b) = 1$ then $r^{-1}(1+b)$ is the inverse of $i\theta_k x$.

We may now extend θ_k to a homomorphism $R \longrightarrow Z_p \otimes R$. If $e' : R \otimes Z_p \longrightarrow Z_p$ is induced by $e : R \longrightarrow Z$ then, for x,y finite-dimensional

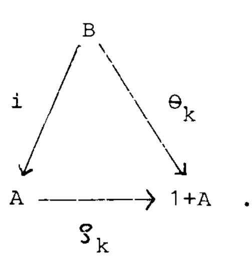
$$e'\theta_k(x-y) = k^{ex-ey} \quad .$$

Therefore θ_k induces a homomorphism

$$\theta_k : B \longrightarrow 1 + A, \qquad A = B \otimes Z_p .$$

<u>Proposition 3.12.2.</u>

<u>The following diagram is commutative:</u>



$(k,p) = 1$

Proof.

Let m = dim x. Then $\gamma_t(x-m)$ is a polynomial of degree $\leq m$. Using

$$\gamma_{t/(t-1)}(x-m)\ \lambda_{-t}(m) = \lambda_{-t}(x)$$

and the definition of θ_k and ρ_k we obtain

$$\theta_k(x) = \rho_k\, i(x-m)\ \theta_k(m)$$

and hence $\theta_k(x-m) = \rho_k\, i(x-m)$. This suffices for the proof.

3.13. Decomposition of p-adic γ-rings.

Let A be a p-adic γ-ring. A fundamental system of neighbourhoods of zero for the p-adic topology may be taken as $(p^nA + A(n) \mid n \geq 1)$. The natural numbers $\mathbb{N}$ are considered as a (dense) subset of the p-adic numbers.

Proposition 3.13.1.

The map

$$\mathbb{N} \times A \longrightarrow A : (k,a) \longmapsto \psi^k(a)$$

is uniformly continuous.

Proof.

Let M = 2N and suppose p^M divides s. If $x_1,\ldots,x_r$ have γ-dimension one then

$$\psi^{k+s}(\Sigma x_i) - \psi^k(\Sigma x_i) = \Sigma (1+x_i)^k((1+x_i)^s-1)$$

$$= p^N S_1 + S_N$$

where S_j is a symmetric function of weight $\geqslant j$ in the x_i for $j = 1,N$. Hence given $N \geqslant 1$ we have shown that there exists $M \geqslant 0$ such that $p^M | s$ implies

$$\psi^{k+s}(x) - \psi^k(x) \in p^N A + A(N)$$

for all x which are a sum of elements of γ-dimension one. By the verification principle for special γ-rings this holds for all x. Hence our map is uniformly continuous in the first variable. Since it is a homomorphism in the second variable it is uniformly continuous.

We can now extend the map $(k,a) \longmapsto \psi^k(a)$ by continuity to a map $Z_p \times A \longrightarrow A$, denoted with the same symbol. Therefore $\psi^k : A \longrightarrow A$ is defined for all $k \in Z_p$ as a continuous homomorphism. Moreover we still have $\psi^k \psi^l = \psi^{kl}$. If Γ denotes the compact topological group of p-adic units then A becomes a topological Γ-module.

By Hensel's Lemma Z_p contains the roots of $x^{p-1} - 1 = 0$. This is a cyclic group of order p-1 generated by d, say. The additive group A splits into eigenspaces of ψ^d

(3.13.2)
$$A = \bigoplus_{i=0}^{p-2} A_i$$
$$A_i = \{ x \in A \mid \psi^d x = d^i x \} .$$

(This is so because A may be considered as $Z_p[C]$ module, where C is the cyclic group generated by T and T acting as ψ^d; and the group algebra $Z_p[C]$ splits completely because Z_p contains the (p-1)-th roots of unity). Since ψ^d is a ring homomorphism we have

(3.13.3)
$$A_i A_j \subset A_{i+j}$$

so that A becomes a Z/(p-1)-graded ring. Let U be the kernel of the reduction mod p $Z_p^* \longrightarrow Z/pZ$. Then U acts on each group A_i because $u \in U$ commutes with ψ^d. Put

$$A_i(n) = A_i \cap A(n). \tag{3.13.4}$$

Then

<u>Proposition 3.13.5.</u>

$A_i(n) = A_i(n+1)$ <u>if</u> $n \not\equiv i \bmod (p-1)$.

<u>Proof.</u> It follows from 3.8.9 that ψ^d acts on $A_i(n)/A_i(n+1)$ as multiplication by d^n. On the other hand, by definition of A_i, it acts as multiplication by d^i. Hence if the quotient is non-zero we must have $n \equiv i \bmod (p-1)$.

<u>3.14. The exponential isomorphism</u> ς_k.

We now come to the main result in the theory of p-adic γ-rings which says that ς_k is an isomorphism if k generates the p-adic units ($p \neq 2$). This is the algebraic reformulation of Atiyah-Tall [14] of the theorem $J'(X) = J''(X)$ of Adams [2] , which is one essential step in the computation of the group J(X) of stable fibre homotopy classes of vector bundles over X.

Let A be a p-adic γ-ring. The group Z_p^* is topologically cyclic if $p \neq 2$. An integer k is a topological generator if and only if k generates $(Z/p^2)^*$.

<u>Theorem 3.14.1.</u>

<u>Let</u> A <u>be</u> <u>a</u> p-<u>adic</u> γ-<u>ring</u> ($p \neq 2$). <u>Assume</u> <u>that</u> $A(n) = A(n+1)$ <u>for</u> $n \not\equiv 0 \bmod p-1$. <u>Let</u> k <u>generate</u> <u>the</u> p-<u>adic</u> <u>units</u>. <u>Then</u>

$$\rho_k : A \longrightarrow 1 + A$$

<u>is an isomorphism</u>.

<u>Proof.</u>

We have A = inv lim A/A(n), 3.8.7. We have a commutative diagram with exact rows (see 3.9.2 and 3.9.3)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(n)/A(n+1) & \longrightarrow & A/A(n+1) & \longrightarrow & A/A(n) & \longrightarrow & 0 \\ & & \downarrow \rho_k & & \downarrow \rho_k & & \downarrow \rho_k & & \\ 0 & \longrightarrow & 1 + A(n)/A(n+1) & \longrightarrow & 1 + A/A(n+1) & \longrightarrow & 1 + A/A(n) & \longrightarrow & 0 \end{array}$$

Therefore it suffices to prove the theorem for A(n)/A(n+1). In that case ρ_k is the map $a \mapsto 1+d(k,n)a$ where $d(k,n) \in Z_p$ is independent of the particular ring, hence is an isomorphism if d(k,n) is a unit. By assumption we only have to consider the case $n \equiv 0 \bmod p-1$. We have computed the numbers d(k,n) in 3.11.2 and it follows from the Clausen-von Staudt Theorem (Borewicz-Safarevic [30] , p. 410) that d(k,n) is a unit in Z_p if k is a p-adic generator and $n \equiv 0$ (p-1), Actually it has been observed by Atiyah-Tall [14] , p. 283 that the results of 3.11 and the Clausen-von Staudt theorem is not necessary. One only needs to produce a p-adic γ-ring such that $A(n)/A(n+1) \neq 0$ for $n \equiv 0$ (p-1) and ρ_k is an isomorphism. We shall describe such an example in a moment and thereby completing the proof of Theorem 3.14.1.

<u>Example 3.14.2.</u>

Let R(Z/p;Q) be the Grothendieck ring of Q[Z/p]-modules. There are two irreducible modules: The trivial module 1, and V which splits as

$W + W^2 + \ldots + W^{p-1}$ over the complex numbers. Hence the augmentation ideal I is the free group on a single generator $x = 1 + W + \ldots + W^{p-1} - p$. By 3.5 the Adams operations are given as follows: $\psi^k = \mathrm{id}$ if $(k,p)=1$, $\psi^k = 0$ if p/k. Evaluation of characters at a generator g of Z/p gives an isomorphism $I \longrightarrow pZ : x \longmapsto -p$. We have

$$\gamma_t(x) = \prod_{i=1}^{p-1} \gamma_t(W^i - 1) = \prod_i ((1-t) + W^i t),$$

and evaluating at g maps the right hand polynomial (short calculation) into $(1-t)^p - (-t)^p$. Therefore $\gamma^r(-p) = 0$ for $r \geq p$ and $p \mid \gamma^r(-p)$ for $1 \leq r \leq p-1$. Since ψ^p acts on I_n/I_{n+1} as multiplication by p^n and $\psi^p = 0$ we see that I_n/I_{n+1} is a p-group (cyclic in this case). Moreover I_n/I_{n+1} is non-zero only if $n \equiv 0\ (p-1)$ because ψ^k, $(k,p)=1$, acts as k^n and as identity. Since $\gamma^{p-1}(-p) = (-1)^{p-1} p$ the lowest power of p attainable in I_n is $(\gamma^{p-1}(-p))^v$ where $(v-1)(p-1) < n \leq v(p-1)$. Hence $I_n/I_{n+1} = Z/p$ for $n \equiv 0\ (p-1)$ and the p-adic topology and the γ-topology coincide. We now compute ρ_k on $I_n/I_{n+1} \otimes Z_p \cong I_n/I_{n+1}$ for $n \equiv 0\ (p-1)$. A generator for I_n/I_{n+1} is the image of p^r. Hence

$$\rho_k(p) = \rho_k(-p)^{-1} = \prod_u ((1 - \tfrac{u}{u-1})^p - (\tfrac{u}{1-u})^p)^{-1}$$

$$= \prod_u \frac{(1-u)^p}{1-u^p} = k^{p-1} = 1 + \frac{k^{p-1}-1}{p} \cdot p$$

Since k generates the p-adic units $m = p^{-1}(k^{p-1}-1)$ is an integer prime to p. We obtain

$$\rho_k(p^r) = \rho_k(p)^{p^{r-1}} = (1+mp)^{p^{r-1}} \equiv 1 + mp^r \bmod p^{r+1}$$

so that ρ_k is on I_n/I_{n+1} the map $\rho_k(a) = 1+ma \in 1 + I_n/I_{n+1}$. Since $I_n/I_{n+1} = Z/p$ this is an isomorphism.

Remark 3.14.3.

We know from 3.11. that for $n = r(p-1)$ ϱ_k in the example above is the map $a \mapsto 1 + (k^n-1)\frac{B_n}{n}a$ and that pB_n is p-integral. We obtain $m \equiv (k^n-1)\frac{B_n}{n} \equiv ((1+mp)^r-1)\frac{B_n}{n} \equiv mrp\frac{B_n}{n} \equiv -m(pB_n) \bmod p$. Hence $pB_n \equiv -1 \bmod p$. This is one of the von Staudt congruences.

We now describe certain instances where the hypothesis of Theorem 3.14.1 is fulfilled.

Let A be any p-adic γ-ring. In 3.13 we have described a splitting of A into eigenspaces A_i of Adams operations ($i = 0,1,\ldots,p-2$). Then ϱ_k induces a map

$$\varrho_k : A_o \longrightarrow 1 + A_o$$

and by 3.13.5 we can apply the Theorem to it:

Proposition 3.14.4.

Let A be a p-adic γ-ring, $p \neq 2$. Let k be a generator of the p-adic units. Then

$$\varrho_k : A_o \longrightarrow 1 + A_o$$

is an isomorphism.

Proposition 3.14.5.

Let A be a p-adic γ-ring. Assume that $\psi^k = id$ for $(k,p) = 1$. Then $A(n)/A(n+1) = 0$ for $n \neq 0\ (p-1)$.

Proof.

For $x \in A(n)/A(n+1)$ we have $x = \psi^k x = k^n x$ and $k^n-1 \in Z_p^*$ for $n \not\equiv 0(p-1)$.

Let A be a p-adic γ-ring. Put

$$(3.14.6)\qquad \begin{aligned} A^{\Gamma} &= \{a \mid \psi^k a = a,\ \text{all } k\} \\ A_{\Gamma} &= A/N,\ N = \{a - \psi^k a \mid a \in A,\ \text{all } k\}. \\ (1+A)^{\Gamma} &= \{1+a \mid \psi^k a = a,\ \text{all } k\} \\ (1+A)_{\Gamma} &= (1+A)/M,\ M = \{(1+a)/\psi^k(1+a) \mid a \in A,\ \text{all } k\}. \end{aligned}$$

Since ρ_k commutes with the Adams operations we have induced maps

$$(3.14.7)\qquad \begin{aligned} (\rho_k)^{\Gamma} &: A^{\Gamma} \longrightarrow (1+A)^{\Gamma} \\ (\rho_k)_{\Gamma} &: A_{\Gamma} \longrightarrow (1+A)_{\Gamma} \end{aligned}$$

Theorem 3.14.8.

If $p \neq 2$ and k is a generator of the p-adic units then the maps 3.14.7

$$(\rho_k)^{\Gamma} \quad \text{and} \quad (\rho_k)_{\Gamma}$$

are isomorphisms.

Proof.

One first shows: If $0 \to X \to Z \to Y \to 0$ is an exact sequence of p-adic γ-rings and the Theorem is true for X and Y, then it is true for Z. The following diagram with exact rows (ker- coker sequences) is commutative

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & X^{\Gamma} & \longrightarrow & Z^{\Gamma} & \longrightarrow & Y^{\Gamma} & \longrightarrow & X_{\Gamma} & \longrightarrow & Z_{\Gamma} & \longrightarrow & Y_{\Gamma} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (1+X)^{\Gamma} & \longrightarrow & (1+Z)^{\Gamma} & \longrightarrow & (1+Y)^{\Gamma} & \longrightarrow & (1+X)_{\Gamma} & \longrightarrow & (1\div Z)_{\Gamma} & \longrightarrow & (1+Y)_{\Gamma} & \longrightarrow & 0. \end{array}$$

One applies the five lemma. (To establish the ker- coker sequence note that

$$0 \longrightarrow X^{\Gamma} \longrightarrow X \xrightarrow{1-\psi^k} X \longrightarrow X_{\Gamma} \longrightarrow 0$$

is exact if k is a generator of the p-adic units). The Theorem is true for A(n)/A(n+1): For $n \not\equiv 0(p-1)$ $A(n)/A(n+1)^{\Gamma} = 0$, $(A(n)/A(n+1))_{\Gamma} = 0$; for $n \equiv 0(p-1)$ ρ_k itself is already an isomorphism by 3.14.1. By the first part of the proof the Theorem is true for all A/A(n). From

$$\text{inv lim } (A/A(n)^{\Gamma}) = (\text{inv lim } A/A(n))^{\Gamma}$$

and an analogous equality for (1+A)/(1+A(n)) the Theorem for A follows. (Note that "invlim" is exact on compact groups.)

We now discuss analogous results for p = 2 where oriented γ-rings are needed. The group of 2-adic units $\Gamma = Z_2^*$ is not (topologically) cyclic, but $\Gamma/\{\pm 1\}$ is; e.g. 3 is a generator. Since $-1 \in Z_p$ the operation ψ^{-1} is defined for p-adic γ-rings, see 3.13.

Proposition 3.14.9.

If A is an oriented p-adic γ-ring then $\psi^{-1} = \text{id}$.

Proof.

If x has γ-dimension 1 then 1+x has λ-dimension 1. Therefore

$$1 = \lambda^0(2+2x) = \lambda^2(2+2x) = \lambda^1(1+x)^2 = (1+x)^2$$

so that $\psi^{-1}(x) = \frac{1}{1+x} - 1 = x$. Hence the Proposition is true for a sum of one-dimensional elements. Now apply a "verification principle".

Theorem 3.14.10.

Let A be an oriented p-adic γ -ring (p any prime). Let k be a generator of $\Gamma / \{\pm 1\}$. Then

$$\rho_k^{or} : A \longrightarrow 1 + A$$

induces isomorphisms

$$(\rho_k^{or})^{\Gamma} \quad \text{and} \quad (\rho_k^{or})_{\Gamma} .$$

If p = 2 then ρ_k^{or} is an isomorphism.

Proof.

Let p = 2. We have to show that A(n)/A(n+1) is mapped isomorphically. By 3.11.5 this group is zero if $n \equiv 1 \bmod 2$. So let n = 2m. Then $\rho_k^{or}(a) = 1 + d'(k,n)a$ and $d'(k,n) = (k^n-1)\frac{B_n}{2n} \in Z_2$ by 3.11.4. In this case if $n = 2^r d$, d odd and $r \geq 1$, then $k^n = 1 + 2^{r+2}c$, c odd, because k is a generator of $Z_2^{*}/\{\pm 1\}$. Hence $(k^n-1)\frac{B_n}{2n} = \frac{c}{d}\, 2B_n$ and by the Clausen-von Staudt theorem $2B_{2m} \equiv -1 \bmod 2$. Therefore $d'(k,n) \in Z_2^{*}$. If one wants to avoid the Clausen-von Staudt theorem one can compute ρ_k^{or} in a special case as in 3.14.2. For $p \neq 2$ $2d'(k,n) = d(k,n) \in Z_p^{*}$ hence $d'(k,n) \in Z_p^{*}$. So one can proceed as in the proof of 3.14.8.

3.15. Thom-isomorphism and the maps θ_k, θ_k^{or}.

Let G be a compact Lie group, $E \longrightarrow X$ a complex G-vector bundle over the compact G-space X. If M(E) is the Thom space of E we have the Thom class $t(E) \in \tilde{K}_G(M(E))$ and $\tilde{K}_G(M(E))$ is a free $K_G(X)$-module with a single generator t(E). Therefore we must have a relation of the type $\psi^k t(E) = \tilde{\theta}_k(E)t(E)$ with a uniquely determined element $\tilde{\theta}_k(E) \in K_G(X)$.

Proposition 3.15.1.

The equality $\Theta_k(E) = \tilde{\Theta}_k(E)$ holds.

Proof. Both Θ_k and $\tilde{\Theta}_k$ are natural for bundle maps and homomorphic from addition to multiplication. By the topological splitting principle it therefore suffices to proof the equality for line bundles E. Let $s^* : \tilde{K}_G(ME) \longrightarrow K_G(X)$ be induced by the zero section. Then $s^* t(E) = 1-E$ and therefore $1-E^k = \psi^k(1-E) = s^* \psi^k t(E) = s^*(\tilde{\Theta}_k(E)t(E)) = \tilde{\Theta}_k(E)(1-E)$. This implies $\Theta_k(E) = 1+E+\ldots+E^{k-1}$ (look e. g. at X a complex projective space). Now use 3.7.2.

For real vector bundles and Θ_k^{or} the situation is analogous but slightly more complicated. We describe the ingredients. Let $E \longrightarrow X$ be a real G-vector bundle of dimension 8n which has a Spin(8n)-structure. With this Spin-structure one defines a Thom-class $t(E) \in \tilde{KO}_G(M(E))$ and the generalized Bott periodicity (Atiyah [10]) says that again $\tilde{KO}_G(M(E))$ is a free $KO_G(X)$-module on $t(E)$. We define $\tilde{\Theta}_k^{or}(E)$ by the equation $\psi^k t(E) = \tilde{\Theta}_k^{or}(E)t(E)$. If k is odd then we also have defined in 3.10 the element $\Theta_k^{or}(E)$ because E, having a Spin-structure, is orientable.

Proposition 3.15.2.

For k odd and E a G-vector bundle with Spin(8n)-structure the equality $\Theta_k^{or}(E) = \tilde{\Theta}_k^{or}(E)$ holds. In particular $\tilde{\Theta}_k^{or}(E)$ is independent of the Spin-structure for odd k.

Proof. Using 3.10.10 a proof is contained in Bott [31], Proposition 10.3, Theorem B on p. 81 and Theorem C" on p. 89.

3.16. Comments.

This section is based on Atiyah-Tall [14]. That paper axiomatizes certain basic results of Adams [1], [2]. The reader should also study the relation-ship between λ-rings, formal groups, Witt-vectors, and Hopf-algebras (Hazewinkel [95]). It would be interesting to investigate the topological significance of the number theoretical properties of the Bernoulli numbers. We also mention the exponential isomorphism for λ-rings obtained in Atiyah-Segal [13] ; this is related to ς_k but gives an isomorphism on the whole ring (under a suitable hypothesis).

3.17. Exercises.

1. Show that the tensor product of special λ-rings A,B is a special λ-ring in a canonical way such that the maps $A \longrightarrow A \otimes B$, $B \longrightarrow A \otimes B$ are λ-homomorphisms.
2. Show that there exists a free special λ-ring U on one generator $u \in U$. This ring is characterized by the following universal property: Given a special λ-ring R and $x \in R$ there is a unique homomorphism $f : U \longrightarrow R$ of λ-rings such that $f(u) = x$.
3. Show that if R is special λ-ring and $x \in R$ n-dimensional then there exists a special λ-ring $S \supset R$ such that $x = x_1 + \ldots + x_n$ where the $x_i \in S$ are one-dimensional (splitting principle).
4. If S is a finite G-set let $\Lambda^i(S)$ be the set of subsets $M \subset S$ with $|M| = i$. The G-action on S induces a G-action on $\Lambda^i(S)$. Show that the $S \longmapsto \Lambda^i(S)$ induce a λ-ring structure on A(G). This structure is in general not special.

4. Permutation representations.

If G is a finite group and S a finite G-set we can consider the associated permutation representation V(S,F) of S over the commutative ring F. The assignment $S \longmapsto V(S,F)$ induces a ring homomorphism

$$h = h_F : A(G) \longrightarrow R(G;F)$$

of the Burnside ring into the representation ring. We shall describe some aspects of this homomorphism in particular when F is a field or the ring of integers Z. We describe the connection to the J-homomorphism of section 2 and to λ-rings.

4.1. p-adic completion.

Let p be a prime number and let G be a p-group. Let

$$A(G)^{\wedge}_p = \mathrm{inv}_n \lim A(G)/p^n A(G) \cong A(G) \otimes_Z Z_p$$

be the p-adic completion of A(G).

If $|G| = p^n$ and $m = q(1,p)$ we have seen in exercise 1.9.4 that $m^{n+1} \subset p\,A(G) \subset m$. Hence

Proposition 4.1.1.

If G is a p-group the p-adic and the m-adic topology on A(G) coincide.

Let now q be a prime different from p. Let $e: R(G,F_q) \rightarrow Z: x \mapsto \dim x$ be the augmentation and $I(G,F_p) = \text{Kernel } e$ the augmentation ideal.

The ring $A(G)^{\wedge}_p$ is a local ring with maximal ideal $m^{\wedge}$, the completion of m.

We now consider the case $p \neq 2$. Since $A(G)[q^{-1}] \subset A(G)^\wedge_p$ we obtain from 2.1 the J-homomorphism

(4.1.2) $$J : R(G,F_q) \longrightarrow A(G)^\wedge_p .$$

We notice that for an F_pG-module V $eJ(V-\dim V) = 1$. Hence

(4.1.3) $$JI(G,F_q) \subset 1 + m^\wedge.$$

The set $1 + m^\wedge \subset A(G)^\wedge_p$ is compact and a topological group with respect to multiplication. A fundamental system of neighbourhoods of 1 is given by $(1+\hat{m}^i)_{i\geq 1}$, or $(1+\hat{m}^i+p^j\hat{m})$. Since

$$J(p^i I(G,F_q)) \subset (1+\hat{m})^{p^i} \subset 1+\hat{m}^{i+1}$$

we see that $J : I(G,F_q) \longrightarrow 1+m^\wedge$ is p-adically continuous and therefore induces a continuous map

(4.1.4) $$J^\wedge : I(G,F_q)^\wedge_p \longrightarrow 1+m^\wedge$$

homomorphic from addition to multiplication.

4.2. Permutation representations over F_q.

We still assume that p is odd and consider the permutation representation map and its p-adic completion

(4.2.1) $$h : A(G) \longrightarrow R(G,F_q)$$
$$h^\wedge : A(G)^\wedge_p \longrightarrow R(G,F_q)^\wedge_p .$$

Since $h(m) \subset p\, R(G,F_q) + I(G,F_q)$ and because the p-adic and $I(G,F_q)$-adic topology on $R(G,F_q)$ coincide (see [6]) we obtain an induced continuous map between multiplicative topological groups

$$(4.2.2) \qquad h^{\wedge}: 1+m^{\wedge} \longrightarrow 1+I(G,F_q)^{\wedge}.$$

<u>Definition 4.2.3.</u>

We call the prime q p-<u>generic</u> if it generates a dense subgroup of the p-adic units (i. e. if q generates Z/p^2Z^*).

<u>Theorem 4.2.4.</u>

<u>Let</u> q <u>be</u> <u>a</u> p-<u>generic</u> <u>prime</u>. <u>Then</u> <u>the</u> <u>composition</u>

$$h^{\wedge}J^{\wedge}: I(G,F_q)^{\wedge} \longrightarrow 1+I(G,F_q)^{\wedge}$$

<u>is</u> <u>an</u> <u>isomorphism</u>.

In fact the proof will show that this is one of the isomorphisms which we had considered in the previous chapter on λ-rings, namely the map ς_q.

<u>Proof.</u>

In order to prove the equality $h^{\wedge}J^{\wedge} = \varsigma_q$ we need only consider cyclic groups $G = Z/p^nZ$ because $J^{\wedge}$, $h^{\wedge}$ and ς_q are compatible with restriction to subgroups and elements in $R(G,F_q)^{\wedge}$ are detected by their restriction to cyclic subgroups.

We begin with the computation of ς_q for $G = Z/p^nZ$. The group algebra $F_qG = F_q[x]/(x^a-1)$, $a = p^n$, decomposes as $\bigoplus_{1\leq t\leq n} F_q[x]/\phi_t(x)$, where $\phi_t(x)$ is the p^t-th cyclotomic polynomial. If q is p-generic then $\phi_t(x)$ is irreducible. Hence the $F_q[x]/\phi_t(x) =: V_t$ are the irreducible

$F_q G$-modules in our case. By 3.12.2 we have the identity

$$\varrho_q(V_t - \dim V_t)\ \theta_q(\dim V_t) = \theta_q(V_t).$$

Over a splitting field F of G the module V_t splits $V_t = \oplus_j V_t(j)$, where $V_t(j)$ is onedimensional and a generator of G acts as multiplication with u^j, where u is a primitive p^t-th root of unity and $j \in Z/p^t Z^*$. Since the θ_q-operations are compatible with field extension we obtain from 3.7.2

$$\theta_q(V_t) = \prod \theta_q(V_t(j)) = \prod (1+V_t(j) + \ldots + V_t(j)^{q-1}) \ .$$

It is enough, by naturality, to study this for t = n. We claim that in $R(G,F) \cong Z[y]/(y^a-1)$ $\theta_q(V_n) = h(1+bG)$ where b satisfies $1+bp^n = q^a$. This means we have to check

$$\prod_j (1 + y^j + \ldots + y^{j(q-1)}) = 1 + b(1 + y + \ldots + y^{a-1}).$$

But this is true if we replace y by a-th roots of unity v and evaluation at such v determines elements of $Z[y]/(y^a-1)$. (This is essentially a computation with modular characters.) Now an easy checking of fixed point dimensions shows that $J(V_n) = 1 + bG$. This shows $hJ(V_t) = \theta_q(V_t)$ and therefore $h^\wedge J^\wedge(V_t - \dim V_t) = \varrho_q(V_t - \dim V_t)$. The equality $h^\wedge J^\wedge = \varrho_q$ is now proved.

We now check that we are in a situation where 3.14.1 and 3.14.5 can be applied. To prove $\psi^k V = V$ for $(k,p) = 1$ and $F_q G$-modules V we again need only consider cyclic G and then this follows from the determination of the irreducible $F_q G$-modules above.

Remark 4.2.5.

If q is p-generic then the decomposition homomorphism

$$d : R(G,Q) \longrightarrow R(G,F_q)$$

(Serre [147] , 15.2) is an isomorphism.

4.3. Representations of 2-groups over F_3.

We now consider the analogue of 4.2 for 2-groups and restrict attention to representations over F_3. We first recall what the theory of oriented γ -rings tells us in this case.

In this section G shall be a 2-group. We have the following objects

$$R(G,F_3) \supset RO(G,F_3) \supset RSO(G,F_3) \supset ISO(G,F_3) .$$

Here $R(G,F_3)$ is the representation ring of F_3G-modules, RO the sub-ring of those modules possessing a G-invariant quadratic form, RSO the subring of F_3G-modules on which each $g \in G$ acts with determinant one, and ISO is the augmentation ideal of zero-dimensional objects.

The ring $RSO(G,F_3)$ is an oriented λ-ring (3.10.2) and $ISO(G,F_3)$ is an oriented γ -ring. Let a roof denote 2-adic completion. We have from 3.14.10

Proposition 4.3.1.

The map

$$\varsigma_3^{or} : ISO(G,F_3)^{\wedge} \longrightarrow 1 + ISO(G,F_3)^{\wedge}$$

is an isomorphism.

In order to relate this isomorphism to the J-homomorphism and to permutation representations we compute the map for cyclic groups $G=Z/2^nZ$. We start with the representation ring.

We have a decomposition of the group ring

$$F_3G \cong \bigoplus_{1\leq t\leq n} F_3[x]/\phi_t(x)$$

where $\phi_t(x)$ is the 2^t-th cyclotomic polynomial. The ϕ_t are no longer irreducible for $t\geq 3$. If $K_t = F_3[u_t]$, where u_t is a primitive 2^t-th root of unity then $[K_t : F_3] = 2^{t-2}$, $t\geq 3$. Moreover $\phi_2(x) = x^2+1$ is irreducible and $K_2 = F_3[u_t] = F_9$.

First assume $t\geq 3$. Let V_t be the F_3G-module K_t where a fixed generator $g\in G$ acts as multiplication with u_t. Then the dual module $V_t^* = \mathrm{Hom}(V_t, F_3)$ is K_t and g acting as u_t^{-1}. Moreover $F_3[x]/\phi_t(x) \cong V_t \oplus V_t^*$ and V_t is not isomorphic to V_t^*. The module V_t cannot carry a G-invariant quadratic form, because this would imply $V_t \cong V_t^*$. But

$$V_t \oplus V_t^* \longrightarrow F_3 : (x,y) \longmapsto \mathrm{Tr}(xy)$$

is a G-invariant, non-degenerate quadratic form (where $\mathrm{Tr} : K_t \longrightarrow F_3$ is the trace map).

If $t = 2$ let $V_t = F_3[u_2] = F_9$ with g acting as multiplication with u_2. Then the norm map $N : F_9 \longrightarrow F_3$ is a G-invariant quadratic form. The associated bilinear form is

$$b : F_9 \times F_9 \longrightarrow F_3 : (x,y) \longmapsto \varphi(x)y + x\,\varphi(y)$$

where φ is the Frobenius automorphism. The determinant of b is one.

Any G-invariant symmetric bilinear form must have determinant one in this case.

Finally there are two one dimensional representations, V_o the trivial representation, and $V_1 = F_3$ with g acting as multiplication with -1. They both carry quadratic forms $q : x \longmapsto x^2$ or $q^- : x \longmapsto -x^2$.

We now enter the computation of ρ_3^{or} for the elements $V_1 - \dim V_1$, $V_2 - \dim V_2$, $V_t + V_t^* - \dim(V_t + V_t^*)$. It is sufficient to compute Θ_3^{or} of the corresponding modules. Since character computations are easier, we compute for QG-module and then use the decomposition homomorphism. Let

$$W_t = \mathbb{Q}[x]/\phi_t(x), \quad t \geq 1$$

with g acting as multiplication with x. Let S_t be the homogeneous G-set with 2^t elements and $V(S_t)$ its permutation representation. Let a_t be the cardinality of K_t. Then we have

<u>Proposition 4.3.1.</u>

<u>For</u> $t \geq 3$:

$$\Theta_3^{or}(W_t) = V(S_1) - V(S_o) + 2^{-t}(a_t - 1)V(S_t).$$

<u>Moreover</u>

$$\Theta_3^{or}(W_2) = V(S_o) - V(S_1) + V(S_2)$$

$$\Theta_3^{or}(W_1 \oplus W_1) = V(S_o) - 2V(S_1) .$$

Proof.

Suppose $t \geq 3$. We compute the character of $\Theta_3^{or}(W_t)$. Over a splitting field W_t decomposes as $W_t = \bigoplus_j (W_t(j) + W_t(-j))$ where $W_t(j)$ is one-dimensional with g acting as multiplication with $(u_t)^j$ and $1 \leq j = 2k + 1 < 2^{t-1}$. From 3.10.12 we obtain

$$\Theta_3^{or}(W_t) = \prod_j (1 + W_t(j) \oplus W_t(-j))$$

with character value at g equal to

$$\prod_j (1 + u^j + u^{-j}), \qquad u = u_t.$$

This product is -1, as can be seen by using the identity

$$\prod_j (x + x^{-1} - (u^j + u^{-j})) = x^{-2^{t-2}} \phi_t(x)$$

and evaluating at x a cubic root of unity. The character value of $\Theta_3^{or}(W_t)$ at non-generators $x \neq 1$ of G is 1. The character value at 1 is a_t. It is an easy matter to check that the permutation representation of $S_1 - S_o + 2^{-t}(a_t - 1)S_t$ has the same character.

Finally $\Theta_3^{or}(W_2) = 1 + W_2$, $\Theta_3^{or}(W_1 \oplus W_1) = 1 + W_1 \oplus W_1$ and the assertion of the proposition is easily verified.

Connecting Θ_3^{or} with the quadratic J-homomorphism and permutation representations presents the difficulty that permutation representations do not generally preserve the orientation. We deal therefore with this problem first.

Let $A_o(G) \subset A(G)$ be the subring generated by finite G-sets S on which each $g \in G$ acts through even permutations.

If S is any finite G-set we can assign to it a homomorphism

$$s(S) : G \longrightarrow Z^* : g \longmapsto \text{signum}(l_g)$$

where $l_g : S \longrightarrow S$ is left translation by g. The assignment $S \longmapsto s(S)$ induces a homomorphism

$$s : A(G) \longrightarrow \text{Hom}(G, Z^*)$$

from the additive group of A(G) into the multiplicative group $\text{Hom}(G,Z^*)$. The kernel of s is $A_o(G)$. Let

$$j : \text{Hom}(G,Z^*) \longrightarrow A(G)$$

be given by

$$j(f) = G/H_f - |G/H_f| + 1$$

where H_f = kernel f. Then j maps into $A(G)^*$. Since $2A(G) \subset$ kernel s everything passes to the 2-adic completions. Let sign be the composition

$$\text{(4.3.3)} \qquad \text{sign} : A(G)^\wedge \xrightarrow[s]{} \text{Hom}(G/Z^*) \xrightarrow[j]{} \dot{A}(G) \subset A(G)^\wedge$$

Then $A(G)^\wedge \longrightarrow A(G)^\wedge : x \longmapsto x + \text{sign}(x) - 1$ has an image in $A_o(G)^\wedge$ and does not change the cardinality.

Let $QS(G,F_3)$ be the monoid of orientation preserving F_3G-modules with quadratic form under orthogonal sum. Denote $f : QS(G,F_3) \longrightarrow ISO(G,F_3)$ the map $(M,q) \longmapsto M - \dim M$.

We define a modified quadratic J-map

$$J' : QS(G,F_3) \longrightarrow A_o(G)^\wedge$$

by $J'(M,q) = (JQ(M,q) + \text{sign } JQ(M,q)-1)_1$ where $(-)_1$ means that we divide the value in the bracket by its cardinality (which is a power of 3, hence invertible in $A_o(G)^\wedge$).

Theorem 4.3.4.

The following diagram is commutative

$$\begin{array}{ccc} QS(G,F_3) & \xrightarrow{\;J'\;} & A_o(G)^\wedge \\ \downarrow f & & \downarrow h \\ ISO(G,F_3) & \xrightarrow{\;\rho_3^{or}\;} & RSO(G,F_3)^\wedge \end{array} .$$

Proof.

It is sufficient to consider cyclic groups $G = Z/2^nZ$. In that case any (M,q) is orthogonal sum of forms carried by one of the modules $V_t + V_t^*$, $t \geq 3$, V_2, $V_1 \oplus V_1$. In the case of $V_t + V_t^*$ the form must be hyperbolic. From 2.3.4 one obtains $JQ(V_t \oplus V_t^*,q) = 1+2^{-t}(a_t-1)S_t$ (compare 4.3.2). Since sign $S_t = S_1-1$ we compute $J'(V_t \oplus V_t,q) = a_t^{-1}(S_1-1+2^{-t}(a_t-1)S_t)$ and with 4.3.2 we obtain the desired commutativity. The remaining cases give the following results:

$JQ(V_2,q) = 1-S_2$, $J'(V_2,q) = \frac{1}{3}(1-S_1+S_2)$

$JQ(V_1 \oplus V_1,q \oplus q) = JQ(V_1 \oplus V_1,q^- \oplus q^-) = 1-2S_1$

$J'(V_1 \oplus V_1,q \oplus q) = \frac{1}{3}(2S_1-1)$

$JQ(V_1 \oplus V_1,q \oplus q^-) = 1+S_1$, $J'(V_1 \oplus V_1,q \oplus q^-) = \frac{1}{3}(2S_1-1)$.

Again with 4.3.2 we obtain the desired commutativity.

4.4. Permutation representations over Q.

The previous investigations can be used to give a very round-about prove of

Theorem 4.4.1.

Let G *be a p-group. Then*

$$h_Q : A(G) \longrightarrow R(G,Q)$$

is surjective.

We make various remarks how this is related to the forgoing results. We have decomposition homomorphisms $d_q : R(G,Q) \longrightarrow R(G,F_q)$ and $d_3 : R(G,Q) \longrightarrow RO(G;F_3)$. If G is a p-group, $p \neq q$ and q is p-generic then d_q is an isomorphism. If G is a 2-group then d_3 is an isomorphism. In order to show that h_Q is surjective one can therefore try to show the same for h_{F_q} or h_{F_3}.

It is now easy to show that the cokernel of h_Q is annihilated by the order of the group G. This can be seen as follows. The characters in R(G,Q) are constant on conjugacy classes and the set of generators of a cyclic group. If $H < G$ is cyclic then $h(G/H)(g)$ is non-zero if and only if g is conjugate to an element in H and $h(G/H)(g) = |G/H^g|$ is divisible by $|NH/H|$. Hence any class function which is constant on generator sets of cyclic groups is a Z-linear combination of $|NH/H|^{-1} h(G/H)$, $H < G$ cyclic. As a consequence h_Q is surjective for a p-group if the p-adic completion is surjective. For $p \neq 2$ this follows immediately from 4.2.4. For p = 2 one deduces from 4.3.4 that

$A_o(G) \longrightarrow RSO(G)$ is surjective. But if V is any $Q[G]$-module let $D(V)$ be its determinant module. Then $D(V) \oplus 1$ is a permutation representation and $V \oplus D(V) \oplus 1$ is orientation preserving. Hence $V = V \oplus D(V) \oplus 1 - D(V) \oplus 1$ is in the image of d_Q.

4.5. Comments.

The material in this section is taken from Segal [146]. The presentation in 4.3 is unsatisfactory; I hope some reader can elaborate on it. There are important connections between the Burnside ring and integral permutation representations, see Oliver [121], [122] and the references there to earlier work of Dress and Endo-Miyata. For 4.4.1 see also Ritter [133].

5. The Burnside-Ring of a Compact Lie Group.

5.1. Euler Characteristic.

We collect the properties of the Euler-Characteristic that we shall need in the sequel and indicate proofs when appropriate references cannot be given.

Let R be a commutative ring and let A be an associative R-algebra with identity (e.g. A = R; A = R[G], G a finite group). In general, an Euler-Poincaré map is a map from a certain category of A-modules to an abelian group which is additive on certain exact sequences. We consider the following sufficiently general situation:

Let $Gr^R(A)$ be the abelian group (Grothendieck group) with generators [M] where M is a left A-module which is finitely generated and projective as an R-module, with relations $[M] = [M'] + [M'']$ for each exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of such modules. Let Gr(A) be the Grothendieck group of finitely generated left A-modules and the analogous relations for exact sequences. A ring R is called regular if it is noetherian and every finitely generated R-module has a finite resolution by finitely generated projective R-modules.

Proposition 5.1.1.

Let R be a regular ring and A an R-algebra which is finitely generated and projective as an R-module. Then the forgetful map $Gr^R(A) \longrightarrow Gr(A)$ is an isomorphism.

Proof.

Swan-Evans [158], p. 2. (The symbol G_0 is used in [158] where we use Gr. Since we do not need G_1 and use G to denote groups we have chosen this non-standard notation.)

Remark.

In the case of the group ring $A = S[\pi]$, S a commutative ring, we denote $Gr^S(A)$ by $R(\pi,S)$. Tensor product over S induces a multiplication and $R(\pi,S)$ becomes a commutative ring the representation ring of π over S.

We call the assignment $M \mapsto [M] \in Gr^R(A)$ a universal Euler-Characteristic for the modules under consideration, because any map $M \mapsto e(M)$, $e(M) \in B$, B an abelian group, such that $e(M) = e(M') + e(M'')$ whenever $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, is induced from a unique homomorphism $h : Gr^R(A) \longrightarrow B$, $e(M) = h[M]$. (Similar definition for Gr(A).) If $R = A$ is a field then $M \longmapsto \dim_R M \in Z$ is such a universal map, establishing $Gr(R) \cong Z$. If $R = A = Z$ then $M \mapsto \text{rank}(M) = \dim_Q(M \otimes_Z Q) \in Z$ is a universal Euler-Characteristic. (by 5.1.1 $Gr^Z(Z) = Gr(Z)$).

If $M : 0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n \longrightarrow 0$ is a complex of A-modules which are finitely generated and projective as R-modules then we define

$$(5.1.2) \qquad \chi(M_\bullet) = \sum_{i=0}^{n} (-1)^i [M_i] \in Gr^R(A)$$

to be the Euler-Characteristic of the complex. We use the same terminology in case of Gr(A). If submodules of finitely A-modules are again finitely generated then for the homology groups $H_i(M_\bullet)$ of a complex

$$\chi(H_\bullet(M_\bullet)) := \quad (M \;) \quad Gr(A).$$

If $0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$ is an exact sequence of complexes then

$$(5.1.4) \qquad \chi(M_\bullet) = \chi(M'_\bullet) + \chi(M''_\bullet)$$

when everything is defined. If one works with $Gr^R(A)$ then one has to use *hereditary rings*, i.e. submodules of projective modules are projective (see Cartan-Eilenberg [45], p. 14 for this notion). Examples are *Dedekind rings* R, i. e. integral domains in which all ideals are projective (see Swan-Evans [158], p. 212 for various characterisations of Dedekind rings).

We now consider the special cases that are relevant for topology. Let (Y,A) be a pair of spaces such that the (singular) homology groups with integral coefficients $H_i(Y,A)$ are finitely generated and zero for large i. Then, by abuse of language, we define the *Euler-Characteristic* $\chi(Y,A)$ *of the pair* (Y,A) to be the integer

$$(5.1.5.) \qquad \chi(Y,A) = \sum_{i \geq 0} (-1)^i \operatorname{rank} H_i(Y,A)$$

with the usual convention $\chi(Y) = \chi(Y,\emptyset)$. Standard properties are (see Dold [75], p. 105):

Proposition 5.1.6.

(i) *If two of the numbers* $\chi(Y)$, $\chi(A)$ *and* $\chi(Y,A)$ *are defined then so is the third, and*

$$\chi(Y) = \chi(A) + \chi(Y,A).$$

(ii) *If* $(Y;Y_1,Y_2)$ *is an excisive triad and if two of the numbers* $\chi(Y_1 \cup Y_2)$, $\chi(Y_1 \cap Y_2)$, $\chi(Y_1) + \chi(Y_2)$ *are defined then so is the third, and*

$$\chi(Y_1) + \chi(Y_2) = \chi(Y_1 \cup Y_2) + \chi(Y_1 \cap Y_2).$$

(iii) *If* (Y,A) *is a relative CW-complex with* Y-A *containing many cells* then $\chi(Y,A)$ is defined and

$$\chi(Y,A) = \sum_{i \geq 0} (-1)^i n_i$$

where n_i is the number of i-cells in $Y - A$.

If F is a field we can consider

$$(5.1.7) \quad \chi(Y,A;F) = \sum_{i \geq 0} (-1)^i \dim_F H_i(Y,A;F),$$

if this number is defined. Then 5.1.6 also holds with this type of Euler-Characteristic.

Proposition 5.1.8.

(i) *If F has characteristic zero then* $\chi(Y,A)$ *is defined if and only if* $\chi(Y,A;F)$ *is defined and* $\chi(Y,A) = \chi(Y,A;F)$.

(ii) *If* $\chi(Y,A)$ *is defined and* (Y,A) *has finitely generated integral homology then* $\chi(Y,A;F)$ *is defined for any field and* $\chi(Y,A) = \chi(Y,A;F)$.

Proof. This is a simple application of the universal coefficient formula. (See Dold [75], p. 156).

One can also define the Euler-Characteristic using various types of cohomology (singular-, Alexander-Spanier-, sheaf-, etc.) and use the universal coefficient formulas to see that homology and cohomology gives the same result under suitable finiteness conditions.

Proposition 5.1.9.

Let $p : E \longrightarrow B$ *be a Serre-fibration with typical fibre F. If* $\chi(B)$ *and* $\chi(F)$ *are defined and the local coefficient system* $(H_*(p^{-1}b;Q))$ *is trivial then* $\chi(E)$ *is defined and*

$$\chi(E) = \chi(F) \quad \chi(B).$$

Proof.

Use the existence of the Serre spectral sequence; apply the Künneth-formula to the E_2-term; use 5.1.3 (see Spanier [152] , p. 481).

We actually need a more general result where fibrations are replaced by relative fibrations and the coefficient system may be non-trivial. This will be done in the next section when a suitable class of spaces with Euler-Characteristic (the Euclidean neighbourhoods retracts) has been described. A really general and satisfactory treatment of the Euler-Characteristic (and its generalization: the Lefschetz number) does not seem to exist.

5.2. Euclidean neighbourhood retracts.

We single out a convenient class of G-spaces X such that for all fixed point sets and other related spaces the Euler-Characteristic is defined.

Let G be compact Lie group. We define a G-ENR (Euclidean Neighbourhood Retract) to be a G-space X which is (G-homeomorphic to) a G-retract of some open G-subset in a G-module V.

Proposition 5.2.1.

If X is a G-ENR and $i : X \longrightarrow W$ a G-embedding into a G-module W then iX is a G-retract of a neighbourhood.

Proof. As in Dold [75] , p. 81, using the Tietze-Gleason extension theorem (Bredon [37] , p. 36; Palais [124] , p. 19).

Proposition 5.2.2.

A differentiable G-manifold with a finite number of orbit types is a G-ENR.

Proof.

Embed the manifold differentiably into a G-module (Wasserman [165]) where it is a retract of a G-invariant neighbourhood.

If we have no group G acting we simply talk about ENR's. The following basic result of Borsuk shows that being an ENR is a local property. Recall that a space X is called locally contractible if every neighbourhood V of every point $x \in X$ contains a neighbourhood W of x such that $W \subset V$ is nullhomotopic fixing x. It is easy to see that an ENR is locally contractible (Dold [75] , p. 81). A space is locally n-connected if every neighbourhood V of every point x contains a neighbourhood W such that any map $S^j \longrightarrow W$, $j \leq n$, is nullhomotopic in V.

Proposition 5.2.3.

If $X \subset \mathbb{R}^n$ is locally (n-1)-connected and locally compact then X is an ENR.

Proof.

Dold [75] , IV 8.12, and 8.13 exercise 4.

Remarks 5.2.4.

A basic theorem of point set topology says that a separable metric space of (covering) dimension $\leq n$ can be embedded in $\mathbb{R}^{2n+1}$; see Hurewicz-Wallman [38] for the notion of dimension and this theorem. Hence a space is an ENR if and only if it is locally compact, separable metric, finite-dimensional and locally contractible. Using a local Hurewicz-theorem (Raußen [131]) one can express the local contractibility in terms of homology conditions.

Proposition 5.2.5.

Let X be a G-ENR. Then the orbit space X/G is an ENR.

Proof.

Let $X \xrightarrow{i} U \xrightarrow{r} X$ be a presentation of X as a neighbourhood retract (i.e. U open G-subset in a G-module, ri = id). We pass to orbit spaces. A retract of an ENR is an ENR. Hence we have to prove the Proposition for X a differentiable G-manifold (and then apply it to the manifold U). Let $p : X \longrightarrow X/G$ be the quotient map. Given $x \in V \subset X/G$, V open, $p^{-1}V$ contains a G-invariant tubular neighbourhood W of the orbit $p^{-1}x$. Hence pW is contractible. Therefore X/G is locally contractible. Moreover X/G is locally compact (Bredon [37], p. 38), separable metric (Palais [124], 1.1.12) and dim $X/G \leq$ dim X (use Hurewicz-Wallman [38]). Now apply 5.2.3, and 5.2.4.

Using 5.2.3 and the following result of Jaworowski we see that being a G-ENR is a local property too.

Proposition 5.2.6.

Let X be a G-space which is separable metric and finite-dimensional. Then X is a G-ENR if and only if X is locally compact, has a finite number of orbit types, and for every isotropy group H < G the fixed point set X^H is an ENR.

Proof.

Jaworowski [102].

Corollary 5.2.7.

If X is a G-ENR then $X_{(H)}$ is a G-ENR for every H < G.

Proposition 5.2.8.

If X is a compact ENR then the Euler-Characteristic $\chi(X)$ *is defined.*

Proof.

X is a retract of a space K which may be given as a finite union of cubes in a Euclidean space. Hence H_iX is a direct summand in H_iK, which is finitely generated and zero for large i.

Proposition 5.2.9.

Let $E \longrightarrow B$ *be a fibre bundle with typical fibre F. If F and B are ENR's then E is an ENR.*

Proof.

Apply 5.2.3.

We now come to the generalization of 5.1.9.

Proposition 5.2.10.

Let $F : (X,A) \longrightarrow (Y,B)$ *be a continuous map between compact ENR's such that* $F(X \setminus A) = Y \setminus B$. *Suppose the induced map* $f : X \setminus A \longrightarrow Y \setminus B$ *is a fibration with typical fibre Z a compact ENR. Then*

$$\chi(X,A) = \chi(Z) \quad \chi(Y,B).$$

The Euler-Characteristic $\chi_c(X \setminus A)$ *of* $X \setminus A$ *computed with Alexander-Spanier cohomology with compact support and coefficients in a field exists and* $\chi(X,A) = \chi_c(X \setminus A)$.

Proof.

Since the integral homology groups are finitely generated, we can compute the Euler-Characteristic using any field of coefficients and homology or cohomology. We use cohomology with Z/2-coefficients. Since ENR's are locally contractible, 5.2.3, we can use singular or Alexander-Spanier cohomology (Spanier [152], 6.9.6.). Using Alexander-Spanier cohomology with compact support we have by Spanier [152], 6.6.11, that

$$H_c^i(X,A) = H_c^i(X \setminus A)$$

and similarly for (Y,B). The fibration $f : X \setminus A \longrightarrow Y \setminus B$ gives us a Leray spectral sequence with E_2-term

$$E_2^{p,q} = H_c^p(Y \setminus B;\ H_c^q(Z))$$

where the coefficients are $H_c^q(Z)$ considered as a local coefficient system on $Y \setminus B$ (Borel [25], XVI. 4.3; [27]). If this local coefficient system is trivial then our assertion follows as in 5.1.9. If it is non-trivial then the following ad hoc argument of Becker and Gottlieb reduces it to the case of a trivial coefficient system: Since $H_c^q(Z)$ is a finite group (Z/2 coefficients!) a finite covering of $Y \setminus B$ will make the coefficient system trivial. The relation $\chi(U') = N\ \chi(U)$ for a finite covering $U' \longrightarrow U$ of degree N (which will be proved in 5.3) and the result for trivial coefficients implies $\chi_c(X \setminus A) = \chi(Z)\ \chi_c(Y \setminus B)$.

Problem 5.2.11.

Give a satisfactory and general (not just for ENR's) proof for 5.2.10 and its generalization to Lefschetz numbers (compare Dold [77]).

Proposition 5.2.11.

Finite G-CW-complexes are G-ENR's.

Proof.

See Illman [100] for the notion of G-CW-complexes.

Use 5.2.3, 5.2.6.

5.3. Equivariant Euler-Characteristic.

If G is a compact Lie group and X is a G-space then the G-action on X induces a G-action on the cohomology groups $H^i(X;M)$ where M is an R-module. If G_o is the component of the identity of G then G_o acts trivially on $H^i(X;M)$ so that $H^i(X;M)$ becomes an $R[G/G_o]$-module. If $H^*(X;M) = (H^i(X;M))_{i\geq o}$ is R-finite, i. e. zero for large i and finitely generated as R-module, then we define the equivariant Euler-Characteristic of the G-space X to be the element

$$(5.3.1) \qquad \chi_G(X;R) = \sum_{i\geq o} (-1)^i \, H^i(X;R) \in Gr(R[G/G_o]).$$

If $R = \mathbb{C}$, the complex numbers, then $\chi_G(X;\mathbb{C}) \in R(G)$, where R(G) denotes the complex representation ring. We use similar definitions for pairs of G-spaces and homology. Actually for general spaces one has to specify the cohomology theory. For simplicity we make the following

Assumption 5.3.2:

X is a G-ENR. Cohomology is Alexander-Spanier cohomology with compact support (in this case isomorphic to sheaf- or presheaf cohomology with compact support; see Spanier [152], Chapter 6; Bredon [35], Chapter III).

Our task in this section is the computation of (5.3.1) in case R is the field of rational numbers. The computation will be in terms of non-equivariant Euler-Characteristics. The reader should convince himself that most of the results to follow are obvious if a finite group acts simplicially on a finite complex. In this case one can compute on the chain level.

Proposition 5.3.3.

Let G be a p-group acting freely on X. Suppose $H^*(X;F_p)$ is F_p-finite. Then $\chi(X/G;F_p)$ is defined and

$$\chi(X;F_p) = |G| \, \chi(X/G;F_p).$$

(*Recall 5.3.2 and that χ is defined using cohomology with compact support.*)

Proof.

If $H \triangleleft G$ then G/H acts freely on the G/H-ENR (by 5.2.5, 5.2.6) X/H. Hence using induction on the order of G it is sufficient to prove the Proposition for G = Z/p. We use the following fact:

(5.3.4) $$H^i(X;F_p) \cong H^i(X/G;A)$$

where A is the local coefficient system (= locally constant sheaf, Spanier [152], p. 360) with stalks $H^0(\pi^{-1}(x);F_p) \cong F_p[G]$, $\pi: X \to X/G$ the quotient map. In our case the group action on $H^i(X;F_p)$ corresponds via 5.3.4 to the group action on the coefficient system, which is a system of $F_p[G]$-modules (for a verification see Floyd [83] , III. 1). Since an $F_p[G]$-module always contains non-trivial G-fixed submodules if G is a p-group (e.g. by 1. 3.) we can find a filtration $A = A_1 \supset A_2 \supset \dots \supset A_k = 0$ of the coefficient system such that A_i/A_{i+1} is the constant system. The Cartan spectral sequence of a covering (Bredon [35] , p. 154) shows $H^i(X/G;F_p)$ to be finite dimensional. From the additivity of the Euler-Characteristic $\chi(X/G;A_i) = \chi(X/G;A_{i+1}) + \chi(X/G;A_i/A_{i+1})$ we obtain the result.

Proposition 5.3.4.

Let the finite group G act freely on X. Suppose $H^*(X;Z)$ is Z-finite. Then $\chi(X/G;Q)$ is defined and

$$\chi_G(X;Q) = \chi(X/G;Q)\cdot Q[G] \in R(G;Q) .$$

(*Here* $Q[G]$ *denotes the regular representation of* G *over* Q.)

Proof.

Two elements of R(G;Q) are equal if their characters are equal. Thus the assertion of the Proposition is equivalent to:

(5.3.5) $$\chi(X) = |G|\,\chi(X/G),$$

(5.3.6) $$\chi_G(X)(g) = 0 \qquad \text{for } g \neq 1.$$

(Note that $\chi_G(X)(g)$ is the Lefschetz-number

$$L(g,X) = \sum_{i\geq 0}(-1)^i\ (\mathrm{Trace}\ (g,H^i(X;Q))$$

for the action of g; and under reasonable circumstances the Lefschetz-number of a map without fixed points should be zero.)

We first prove 5.3.5 and 5.3.6 for cyclic groups. Since $H^*(X;Z)$ is finite the universal coefficient formula for cohomology with compact support (Spanier [152] , p. 338) shows

(5.3.7) $$\chi(X;Q) = \chi(X;F_p).$$

The Cartan spectral sequence of a covering shows that $H^*(X/G;Z)$ is Z-finite. Hence we obtain from 5.3.3 and 5.3.7, using induction on $|G|$, that 5.3.5 is true for cyclic G.

The existence of the transfer for finite groups implies the isomorphism (Bredon [37] , III 7.2)

(5.3.8) $$H^i(X,\mathbb{Q})^G \cong H^i(X/G;\mathbb{Q}) \ .$$

Since for any character ψ of G dim $\psi^G = |G|^{-1} \sum \psi(g)$ we obtain from 5.3.5 and 5.3.8

(5.3.9) $$\sum_{g\neq 1} \chi_G(X)(g) = 0 \ .$$

Using this we prove 5.3.6 for cyclic groups by induction over the group order: We start with

$$H^i(X,\mathbb{C}) \cong H^i(X/G;A)$$

where A again is the local coefficient system with typical stalk $\mathbb{C}[G]$. Let g be a generator of G. We decompose the coefficient system A according to the irreducible $\mathbb{C}[G]$-modules

$$A = \oplus A_j \ , \quad 0 \leq j < m = |G|$$

where g acts on A_j through multiplication with $\zeta^j = \exp(2\pi ij/m)$. The equalities

$$\begin{aligned} Tr(g^k,H^i(X;\mathbb{C})) &= \sum_j Tr(g^k,H^i(X/G;A_j)) \\ &= \sum_j \zeta^{jk} \dim H^i(X/G;A_j) \end{aligned}$$

yield for the Lefschetz-number

$$L(g^k,X) = \Sigma_j \; \zeta^{jk} \; \chi(X/G;A_j).$$

But $L(g^k,X) \in Z$ is obtained from $L(g,X)$ for $(k,m) = 1$ by applying a Galois automorphism of $Q(\zeta)$ over Q. Therefore $L(g^k,X) = L(g,X)$ for $(k,m) = 1$. From 5.3.9 we obtain

$$(5.3.10) \qquad 0 = \sum_{(k,m)=1} L(g^k,X) + \sum_{(k,m)\neq 1} L(g^k,X)$$

By the inductive assumption the second sum in 5.3.10 is zero, and since the summands of the first sum are all equal we see that $L(g,X) = 0$. This proves 5.3.6 in general. Again using 5.3.8 and 5.3.9 we obtain 5.3.5 for general G.

We have actually proved in 5.3.4 a special case of the Lefschetz fixed point theorem.

<u>Proposition 5.3.11.</u>

<u>Let</u> X <u>be a compact</u> G-ENR <u>where</u> G <u>is a cyclic group with generator</u> g. <u>Then the Lefschetz number</u>

$$L(g,X) = \Sigma_{i\geq 0}(-1)^i \text{ Trace } (g,H^i(X;Q))$$

<u>is equal to the Euler-Characteristic</u> $\chi(X^g)$.

<u>Proof.</u>

Let $X_1 = X^g$, $X_2,\ldots,X_r$ be the orbit bundles of X. Then $H^*(X_i,Z)$ (cohomology with compact support) is Z-finite and $L(g,X_j) = 0$ for $j > 1$ by 5.3.4. Hence

$$L(g,X) = \Sigma_j L(g,X_j) = L(g,X_1)$$

and clearly $L(g,X_1) = \chi(X^g)$.

<u>Corollary 5.3.12.</u>

Let G be a finite group and let X be a compact G-ENR. Then

$$\chi(X/G) = |G|^{-1} \sum_{g\in G} \chi(X^g).$$

<u>Proof.</u>

From $H^i(X/G) \cong H^i(X)^G$ and $\dim H^i(X)^G = |G|^{-1} \sum_{g\in G} \mathrm{Trace}\,(g,H^i(X))$ the result follows, using 5.3.11.

We can now compute the equivariant Euler-Characteristic $\chi_G(X)$.

<u>Theorem 5.3.13.</u>

Let G be a compact Lie group and X be a compact G-ENR. Then

$$\chi_G(X) = \sum_{(H)} \chi(X_{(H)}/G)\ \chi_G(G/H)$$

where the sum is taken over those isotropy types (H) of X such that NH/H is finite.

<u>Proof.</u>

By additivity of the Euler-Characteristic

$$\chi_G(X) = \sum_{(H)} \chi_G(X_{(H)}).$$

Thus we have to show: $\chi_G(X_{(H)}) = 0$ if NH/H is infinite and

(5.3.14) $$L(g,X_{(H)}) = \chi(X_{(H)}/G)\ L(g,G/H)$$

otherwise ($g \in G$). Let C be the closed subgroup of G generated by g.

Since $L(g,Y)$ only depends on the image of g in the group of components of C we can find an element $h \in C$ of finite order such that $L(g,Y)=L(h,Y)$ for all Y. We fix h with this property. Since X is a compact G-ENR we can find compact G-ENR's $Y \supset Z$ in X such that $Y \setminus Z = X_{(H)}$. The proof of 5.3.11 shows

$$L(h,X_{(H)}) = \chi(X_{(H)}{}^h).$$

Using the fibre bundle

$$G/H \longrightarrow X_{(H)} \longrightarrow X_{(H)}/G$$

and 5.2.10 we obtain

$$\chi(X_{(H)}{}^h) = \chi(X_{(H)}/G) \quad \chi(G/H^h).$$

Again by 5.3.11 $\chi(G/H^h) = L(g,G/H)$, so we see that 5.3.14 is true in general. But $\chi(G/H^h) = 0$ if NH/H is infinite because NH/H acts freely on G/H^h.

Remark 5.3.15.

If G is finite then $\chi_G(G/H)$ is just the permutation representation associated to the G-set G/H. In general $\chi_G(G/H) \in R(G/G_o;\mathbb{Q})$ where G_o is the component of the identity of G. We would like to see that this is actually a permutation representation.

Problem 5.3.16.

What are the most general assumptions on the spaces which imply the decomposition formula 5.3.13? A similar formula holds for the equivariant Lefschetz number of a G-map $f : X \longrightarrow X$ between compact G-ENR's. Also this should be generalized to more general spaces.

5.4. Universal Euler-Characteristic for G-spaces.

The classical computation of the Euler-Characteristic from a cell decomposition of a space indicates that suitable axioms (like 5.16 (i), (ii)) determine the Euler-Characteristic uniquely. This is carried out in Watts [166] . We present a similar argument for G-spaces without insisting on a minimal set of axioms.

An Euler Characteristic for finite G-CW-*complexes* consists of an abelian group A and map b which associates to each finite CW-complex X an element $b(X) \in A$ such that:

(i) If X and Y are G-homotopy-equivalent then $b(X) = b(Y)$.

(ii) If X and Y are subcomplexes of Z then

$$b(X) + b(Y) = b(X \cup Y) + b(X \cap Y).$$

Given such an Euler-Characteristic b we show

Proposition 5.4.1.

Let X *be a finite* G-CW-*complex. Then*

$$b(X) = \Sigma_{(H)} \; n_H \; b(G/H)$$

where

$$n_H = \Sigma_{i \geq 0} \; (-1)^i \; n(H,i)$$

n(H,i) *the number of* i-*cells of type* (H), *and the sum is taken over conjugacy classes of subgroups of* G.

Proof.

Induction on the number of cells and dimension. Let $Z = X \cup (G/H \times e^n)$ be obtained from X by attaching an n-cell of type (H). Let $Y = G/H \times D^n(1/2)$

be the closed cell in $G/H \times e^n$ of radius 1/2. If we remove Y from Z then the resulting space is G-homotopy-equivalent to X. Therefore

$$b(Z) = b(X) + b(G/H \times D^n) - b(G/H \times S^{n-1}).$$

One shows by induction

$$b(G/H \times S^n) = 1 + (-1)^{n+1}\, b(G/H);$$

namely if D_+ and D_- are the upper and lower hemisphere of S^n respectively then

$$b(G/H \times S^n) = b(G/H \times D_+) + b(G/H \times D_-) - b(G/H \times S^{n-1})$$

$$= 2b(G/H) - (1+(-1)^n)\, b(G/H)$$

$$= 1 + (-1)^{n+1}\, b(G/H).$$

Put together we obtain

$$b(Z) = b(X) + (-1)^n\, b(G/H),$$

the induction step.

An <u>Euler-Characteristic</u> (U(G),u) for finite G-CW-complexes is called <u>universal</u>, if every Euler-Characteristic (A,b) as above is obtained from (U(G),u) by composing with a unique homomorphism $U(G) \longrightarrow A$. As usual for universal objects uniqueness up to isomorphism follows.

From 5.4.1 we obtain <u>existence:</u>

(5.4.2) U(G) free abelian group with basis

$$[G/H] \, , \; (H) \in C(G).$$

$$u(X) = \sum_{(H)} n_{(H)} [G/H] .$$

Instead of u(X) we also write $[X]$, in accordance with the notation $[G/H]$ for the basis elements. We now aim at another characterisation of U(G) which is not based an CW-complex and which shows that b(X) in 5.4.1 is independent of the cell decomposition.

<u>Proposition 5.4.3.</u>
<u>We have</u> $[X] = [Y]$ <u>in</u> U(G) <u>if and only if for all</u> $H < G$

$$\chi(X^H/NH) = \chi(Y^H/NH).$$

<u>Proof.</u>
Suppose $[X] = [Y]$. We consider the mapping

$$b_H : Z \longmapsto \chi(Z^H/NH)$$

from finite G-CW-complexes into Z. This mapping satisfies (i) and (ii) in the definition of an Euler-Characteristic for finite G-CW-complexes. From the universal property of U(G) we obtain $b_H(X) = b_H(Y)$. For the converse be have to show that the totallity of maps $b_H : U(G) \longrightarrow Z$ defines an injective map $U(G) \longrightarrow \prod_{(H)} Z$. Let $0 \neq x = \sum a_H [G/H] \in U(G)$. Let H be maximal such that $a_H \neq 0$. Then

$$b_H(x) = a_H \; \chi((G/H^H)/NH) = a_H \neq 0.$$

We now redefine the group U(G).

Definition and Proposition 5.4.4.

On the set of compact G-ENR introduce the equivalence relation: $X \sim Y \iff$ for all $H < G$ the equality $\chi(X^H/NH) = \chi(Y^H/NH)$ holds. Let $U(G)$ be the set of equivalence classes and let $[X] \in U(G)$ be the class of X. Disjoint union induces on $U(G)$ the structure of an abelian group. This group is free abelian with basis $[G/H]$, $H \in C(G)$. We have

$$(5.4.5) \qquad [X] = \Sigma_{(H)} \chi_c (X_{(H)}/G) [G/H] .$$

Proof.

We have to show that inverses exist for addition. Let K be a compact ENR with trivial G-action and $\chi(K) = -1$. Then $[X] + [K \times H] = 0$ in $U(G)$ because $\chi(X^H) + \chi((K \times H)^H) = 0$ for all $H < G$. As in the proof of 5.4.3 one shows that the $[G/H]$ are linearly independent. We show that the $[G/H]$ span $U(G)$ by proving 5.4.5. By additivity of the Euler-Characteristic we have

$$\chi(X^K/NK) = \Sigma_{(H)} \chi_c(X_{(H)}{}^K/NK).$$

Now $X_{(H)} \longrightarrow X_{(H)}/G$ is a fibre bundle with fibre G/H and as G-space $X_{(H)}$ has the form $G/H \times_{NH} X_H$ (see Bredon , p. 88). Hence $X^K_{(H)}/NK \longrightarrow X_{(H)}/G$ is a fibre bundle with fibre $G/H^K/NK$. From we obtain

$$\chi_c(X_{(H)}{}^K/NK) = \chi((G/H^K)/NK) \quad \chi_c(X_{(H)}/G).$$

This shows that both sides of 5.4.5 describe the same element in $U(G)$.

Definition and Proposition 5.4.6.

Cartesian product of representatives induces a multiplication on $U(G)$. Addition and multiplication make $U(G)$ into a commutative ring with identity. This ring is called the Euler-ring of the compact Lie group G.

Proof.

We need only show that multiplication is well-defined, i. e. we have to show that the numbers $\chi(X \times Y)^K/NK)$ can be computed from the $\chi(X^H/NH)$, $\chi(Y^H/NH)$ or, equivalently, from the $\chi_c(X_H/NH)$, $\chi_c(Y_H/NH)$. We begin with

$$\chi((X \times Y)^K/NK) = \Sigma_{(H)} \chi_c(X \times Y_{(H)})^K/NK).$$

The map

$$(X \times Y_{(H)})^K/NK \longrightarrow Y_{(H)}/G$$

is a fibre bundle with fibre

$$(X^K \times G/H^K)/NK.$$

Now we use the fact that G/H^K consists of a finite number of NK-orbits (Bredon [37] , p. 87), say

$$G/H^K = \Sigma_U NK/U$$

as NK-space. Using this information and 5.2.10 we obtain

$$\chi_c((X \times Y_{(H)})^K/NK) = \Sigma_U \chi_c(Y_{(H)}/G) \quad \chi(X^K/U).$$

Finally, using

$$\chi(X^K/U) = \Sigma_{(H)} \chi_c(X_{(H)}/G) \chi((G/H^K)/U),$$

we see that $\chi((X \times Y)^K/NK)$ can be computed from the $\chi_c(X_H/NH)$, $\chi_c(Y_H/NH)$.

We show in the next section that for finite G U(G) is the Burnside ring of G. For non-finite G U(G) contains nilpotent elements. In order to obtain the product structure one has to compute $[G/H]\ [G/K]$.

Proposition 5.4.7.

Suppose NH/H is not finite. Then $[G/H] \in U(G)$ is nilpotent.

Proof.

By the descending chain condition for subgroups of G the spaces G/H^k, $k \geq 1$, altogether only contain a finite number of isotropy groups. If $[G/H]^k = \sum_{(K)} a_K\ [G/K]$ with $a_K \neq 0$ and (K) maximal with this property then $[G/H]^{k+1}$ does not contain $[G/K]$ with a non-zero coefficients: Expanding $[G/H]^{k+1}$ then G/K could only occur from the expansion of $a_K\ [G/H]\ [G/K]$. But $(G/H \times G/K)_K = G/H^K \times NK/K$ and therefore $\chi_c((G/H \times G/K)_K/NK) = \chi(G/H^K) = 0$ because NH/H acts freely on G/H^K and $\chi(NH/H)$ is zero if NH/H is not finite (e.g. because a circle group acts freely on NH/H).

5.5. The Burnside ring of a compact Lie group.

Let G be a compact Lie group. On the set of compact G-ENR's consider the equivalence relation: $X \sim Y \Leftrightarrow$ for all $H < G$ the Euler-Characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal. Let A(G) be the set of equivalence classes and let $[X] \in A(G)$ be the class of X. Disjoint union and cartesian product induce composition laws addition and multiplication, respectively, on A(G). It is easy to see that A(G) with these composition laws is a commutative ring with identity. We call A(G) the Burnside ring of G. We will show in a moment that this definition is consistent with the earlier one of section 1. (for finite G).

Let $\phi(G)$ be the set of conjugacy classes (H) such that NH/H is finite.

Proposition 5.5.1.

Additively, A(G) is the free abelian group on [G/H], (H) ∈ Φ(G). For a compact G-ENR X we have the relation

$$[X] = \Sigma_{(H)\in\Phi(G)}\ \chi(X_{(H)}/G)\ [G/H]\ .$$

The assignment $X \longmapsto \chi(X^H)$ *induces a ring homomorphism* $\varphi_H : A(G) \longrightarrow Z$.

Proof. (Compare 5.4.4).

The last assertion is obvious from the definition. The [G/H], (H) ∈ Φ(G), are linearly independent: Given $x = \Sigma\ a_H\ [G/H] \in A(G)$. Choose (H) maximal such that $a_H \neq 0$. Then

$$\varphi_H\ x = a_H\ \chi(G/H^H) = a_H\ |NH/H| \neq 0$$

and therefore $x \neq 0$.

Given a comapct G-ENR X. Then

$$\chi(X^K) = \Sigma_{(H)}\ \chi(X_{(H)}{}^K) = \Sigma\ \chi(G/H^K)\ \chi(X_{(H)}/G).$$

The summands with NH/H not finite vanish, because NH/H then acts freely on G/H^K so that $\chi(G/H^K) = 0$. This computation shows that [X] and $\Sigma\ \chi(X_{(H)}/G)\ [G/H]$ have the same image under φ_K, (K) ∈ Φ(G), hence are equal in A(G).

The map

$$v : U(G) \longrightarrow A(G) : [X] \longrightarrow [X]$$

is a well-defined ring homomorphism. By 5.5.1 and 5.4.4 it is surjective, and bijective for finite G. In particular we have

Proposition 5.5.2.

For finite G the rings U(G), A(G), and the Burnside ring of finite G-sets are canonically isomorphic.

Proposition 5.5.3.

The kernel of v : U(G) ⟶ A(G) is the nilradical (= set of nilpotent elements) of U(G).

Proof.

Since the φ_H : A(G) ⟶ Z detect the elements of A(G) the ring A(G) cannot have nilpotent elements (different from zero). Now use 5.4.4, 5.4.7, and 5.5.1.

Remark 5.5.4.

The previous Proposition implies in particular that U(G) and A(G) have the same prime ideal spectrum.

Remark 5.5.5.

In contrast to the situation in section 1 with our new definition of A(G) also the negatives of all elements are represented by geometric objects.

We now give some immediate applications of the geometric definition of A(G).

Recall that we have in 5.3 associated with every compact G-ENR X the equivariant Euler-Characteristic

$$\chi_G(X) = \sum_{i \geq 0} (-1)^i \, H^i(X;Q) \in R(G;Q).$$

Proposition 5.5.6.

The assignment $X \longmapsto \chi_G(X)$ induces a ring homomorphism

$$\chi_G : A(G) \longrightarrow R(G;Q).$$

Proof. In order to show χ_G is well-defined we have to show that the character $\chi_G(X)$ can be computed from Euler-Characteristics of fixed point sets. But this is the content of 5.3.11, and the same Proposition shows that χ_G respects addition and multiplication.

Remark 5.5.7.

The homomorphism χ_G generalizes the permutation representation of finite G-sets.

We have mentioned in 1.5 the construction of units of A(G) using representations. We can now make this precise.

The homomorphisms $\varphi_H : A(G) \longrightarrow Z$ combine to an injective (by definition A(G)) ring homomorphism

(5.5.8) $$\varphi : A(G) \longrightarrow \prod_{(H)} Z$$

where the product is taken over the set C(G) of conjugacy classes of closed subgroups of G. We use φ to identify elements of A(G) with functions $C(G) \longrightarrow Z$ (see 5.6 for an elaboration of this point of view).

Proposition 5.5.9.

Let V be a real representation of G. Then $u(V) : (H) \longmapsto (-1)^{\dim V^H}$ is a unit of A(G). The assignment $V \longmapsto u(V)$ induces a homomorphism

$$u : R(G;\mathbb{R}) \longrightarrow A(G)^*.$$

(*Here* $R(G;\mathbb{R})$ *is the real representation ring of* G, *also denoted* RO(G).)

Proof.

Let S(V) be the unit sphere in V. Then

$$\chi(S(V)^H) = 1 - (-1)^{\dim V^H}.$$

Hence $1 - [SV] \in A(G)$ represents the function u.

Proposition 5.5.10.

The multiplication table of the $[G/H] \in A(G)$ *has non-negative coefficient, i.e. if*

$$[G/H]\ [G/K] = \Sigma_{(L)}\ n_L\ [G/L]$$

then $n_L \geq 0$.

Proof.

We have $n_L = \chi((G/H \times G/K)_{(L)}/G)$.

Moreover

$$(G/H \times G/K_{(L)}/G \cong (G/H \times G/K)_L/NL$$

$$\subset (G/H \times G/K)^L/NL .$$

But by Bredon [37] , II. 5.7, the space G/H^L consists of finitely many NL/L-orbits. Since NL/L is finite the set $(G/H \times G/K)_{(L)}/G$ is finite and its Euler-Characteristic therefore non-negative.

5.6. The space of subgroups.

We recall some notions from point set topology. Let E be a metric space with bounded metric d. Let F(E) be the set of non-empty subsets of E.

On $F(E)$ one has the <u>Hausdorff metric</u> h defined by

$$h(A,B) = \max(r(A,B),\ r(B,A))$$

with
$$r(A,B) = \sup(d(x,B) \mid x \in A).$$

If E is complete then $F(E)$ is complete. If E is compact then $F(E)$ is compact.

The convergence of a sequence X_i to the limit X can be expressed as follows: For any $\varepsilon > 0$ there exists n_o such that for $n > n_o$:

(a) for $x \in X_n$ there exists $y \in X$ with $d(x,y) < \varepsilon$.

(b) for $x \in X$ there exists $y \in X_n$ with $d(x,y) < \varepsilon$.

If Y_n is the closure of $\cup_{p \geq o} X_{n+p}$ then X is the intersection of the Y_n.

We want to use this metric on the set $S(G)$ of closed subgroups of the compact Lie group G.

<u>Proposition 5.6.1.</u>

(i) $S(G)$ <u>is a closed (hence compact) subset of</u> $F(G)$.

(ii) <u>The action</u> $G \times S(G) \longrightarrow S(G) : (g,H) \longmapsto gHg^{-1}$ <u>is continuous. The quotient space</u> $C(G)$ <u>is a countable, hence a totally disconnected, compact Hausdorff space</u>.

(iii) $\Phi(G) \subset C(G)$ <u>is a closed subspace</u>.

<u>Proof.</u>

(i) We start with a bi-invariant metric d on G. Let $X = \lim H_i$, $H_i \in S(G)$. Given $x,y \in X$, $\varepsilon > 0$, choose n_o such that for $n > n_o$ there exists x_n, $y_n \in H_n$ with $d(x,x_n) < \varepsilon/2$, $d(y,y_n) < \varepsilon/2$. Then $d(xy^{-1}, x_n y_n^{-1}) < \varepsilon$. If $xy^{-1} \notin X$ then $X \cup \{xy^{-1}\}$ would satisfy conditions (a), (b) above, a contradiction.

(ii) Let $\lim g_i = g$ in G and $\lim H_i = H$ in S(G). Using $d(g_n x_n g_n^{-1}, gxg^{-1}) \leq 2d(g,g_n) + d(x,x_n)$, which follows from the triangle inequality and bi-invariance, one shows that gHg^{-1} is precisely the set of points satisfying (a) and (b) above for the sequence $g_n H_n g_n^{-1}$. The space C(G) is countable: see Palais [124], 1.7.27.

(iii) We show that $S_o(G) = \{H | NH/H \text{ finite}\}$ is closed in S(G). Let $H = \lim H_i$, $H_i \in S(G)$. By a theorem of Montgomery and Zippin (Bredon [37], p. 87) there exists an $\varepsilon > 0$ such that any subgroup in the ε-neighbourhood of H is conjugate to a subgroup of H. Hence the H_i are eventually conjugate to subgroup of H. But if $K \in S_o(G)$ and $K < H$ then $H \in S_o(G)$; this follows e.g. from Bredon [37], II. 5.7, because G/H^K consists of finitely many NK/K-orbits hence is a finite set with free NH/H-action.

We now show that convergence in S(G) and C(G) is equivalent in the following sense.

Proposition 5.6.2.

Let (H) = lim (H_i) in C(G). There exists an n_o and $K_n \in S(G)$, $n \geq n_o$, such that $(K_n) = (H_n)$, $K_n < H$, $\lim K_n = H$.

Proof.

By the theorem of Montgomery and Zippin (Bredon [37], II. 5.6) we can find for each $\varepsilon > 0$ an integer $n_o(\varepsilon)$ such that for $n > n_o(\varepsilon)$ there exists an u_n with $d(u_n,1) < \varepsilon$ and $u_n H_n u_n^{-1} < H$. Therefore we can find a sequence $g_n \in G$ converging to 1 such that for almost all n $g_n H_n g_n^{-1} < H$.

In view of the preceding Proposition it is interesting to know which compact Lie groups G are limits of a sequence of proper subgroups.

<u>Proposition 5.6.3.</u>

<u>G is a limit of proper subgroups if and only if G is not semi-simple.</u>

<u>Proof.</u>

Suppose $G = \lim H_n$, $H_n \neq G$. Let G^o be the component of 1 of G and put $K_n = G^o \cap H_n$. Then $\lim K_n = G^o$ so that without loss of generality we can assume that G is connected. By passing to a subsequence we can assume that the components H_n^o converge to H and therefore must have eventually the same dimension as H. But then the H_n^o are conjugate to H and by conjugating the whole sequence we arrive at the situation: $G = \lim L_n$, $L_n^o =: L$ for all n, $L \neq G$. Since $L \triangleleft L_n$ we must have $L \triangleleft G$ and G/L is the limit of finite subgroups L_n/L. We now invoke the theorem of Jordan (Wolf [163]) which says that there exists an integer j such that any finite subgroup of G/L has a normal abelian subgroup of index less than j. Choose such a large abelian normal subgroup A_n in L_n/L. The limit A of the A_n is then an abelian normal subgroup of index less than j in G/L. Since G/L is connected we must have G/L = A a torus and therefore G is not semi-simple.

Conversely if G is not semi-simple we can find a normal subgroup L of G^o such that G^o/L is a non-trivial torus (Hochschild [97] , XIII Theorem 1.3). By Lie algebra considerations (e.g. Helgason [96] , II. Proposition 6.6) the group L is a characteristic subgroup of G^o and therefore a normal subgroup of G. Therefore G/L =: P is a finite extension of a torus

$$1 \longrightarrow T \longrightarrow P \longrightarrow F \longrightarrow 1,$$

T a torus, F finite. If we show that P is a limit of proper subgroups then G is a limit of proper subgroups. We shall show in section 5.10 what the finite subgroups of P are, in particular we shall see that P is a limit of finite subgroups.

Proposition 5.6.4.

If X is a compact G-ENR then the mapping $C(G) \longrightarrow Z : (H) \longmapsto \chi(X^H)$ is continuous (Z carries the discrete topology).

Proof.

Let $(H) = \lim (H(i))$. By 5.6.2 we can assume $H(i) < H$ and $H = \lim H(i)$. We can and do assume $H = G$ (otherwise consider the H-space X). We choose a bi-invariant metric on X. Put $\varepsilon = \min h(K,G)$ where (K) runs through the finite set of orbit-types of X unequal to G. Since $(L) < (K)$ implies $h(L,G) \geq h(K,G)$ we have: $h(L,G) < \varepsilon$ implies $(L) \not< (K)$ for all isotropy types of X except possibly (G). Thus if $h(H(i),G) < \varepsilon$ then $X^{H(i)} = \cup X^{H(i)}_{(K)} = X^{H(i)}_{(G)} = X^G$.

5.7. The prime ideal spectrum of A(G).

Recall the ring homomorphisms $\varphi_H : A(G) \longrightarrow Z$ (see 5.5). If $(p) \subset Z$ is a prime ideal then

$$q(H,p) := \varphi_H^{-1}(p) \subset A(G)$$

is a prime ideal of A(G). We show that all prime ideals of A(G) arise in this way.

Proposition 5.7.1.

Given $H \triangleleft K < G$. Assume that K/H is an extension of a torus by a finite p-group (K/H a torus if p = o). Then $q(H,p) = q(K,P)$.

Proof.

For a certain L we have $H \triangleleft L \triangleleft K$, L/H is a torus, and K/L a finite p-group. Let X be a compact G-ENR. The group K/L acts on M^L with fixed point set M^K. Hence $\chi(M^K) \equiv \chi(M^L) \bmod p$ and $\chi(M^L) = \chi(M^H)$ by an easy application of Theorem 5.3.

Theorem 5.7.2.

Every prime ideal q of A(G) has the form q(H,p) for a suitable (H) $\in \Phi(G)$. Given q there exists a unique (K) $\in \Phi(G)$ with q = q(K,p) and $\varphi_K(G/K) \not\equiv 0(p)$ where p is the characteristic of A(G)/q.

Proof.

We closely follow Dress [79] ! Let

$$T(q) = \{ (H) \in \Phi(G) \mid [G/H] \notin q \} .$$

Then T(q) is not empty because (G) $\in \Phi(G)$ and $[G/G] = 1 \notin q$. Let (H) be minimal in T(q); this exists because compact Lie groups satisfy the descending condition. We claim that for any $x \in A(G)$ we have a relation of the type

$$(5.7.3) \qquad [G/H]\, x = \varphi_H(x)\, [G/H] + \sum a_K\, [G/K]$$

where the sum is over (K) < (H), (K) $\neq$ (H). To see this we take $x = [X]$ look at the orbits of $G/H \times X$ and see from 5.5.1 that a relation must hold as claimed with some constant c instead of $\varphi_H(x)$: We then determine c if we apply φ_H to both sides of this equation. (This uses $\varphi_H(G/H) \neq 0$, i.e. (H) $\in \Phi(G)$.) But 5.7.3 implies $[G/H]\, x \equiv \varphi_H(x)[G/H]$ mod q (by minimality of G/H) and dividing by $[G/H] \notin q$ we get $x \equiv \varphi_H(x)$ mod q or q = q(H,p) with p the characteristic of A(G)/q.

If K is any subgroup of G with q = q(K,p) and $\varphi_K(G/K) \not\equiv 0 \bmod p$ for p = char A(G)/q then for an (H) as in the beginning of the proof $\varphi_K(G/K) \equiv \varphi_H(G/K) \not\equiv 0 \bmod p$. In particular G/H^K is not empty; and similarly G/K^H is not empty. This can only happen if $(H) = (K)$.

Proposition 5.7.4.

Every homomorphism $f : A(G) \longrightarrow R$ *into an integral domain* R *has the form* $f(x) = \varphi_K(x)\cdot 1$ *for a suitable* $K < G$.

Proof.

The kernel of f is a prime ideal $q(K,p)$. Therefore $f : A(G) \longrightarrow A(G)/q(K,p) \longrightarrow R$ must be the map $x \longmapsto \varphi_K(x)\cdot 1$, because there is a unique isomorphism $A(G)/q(K,p) \cong Z/(p)$.

Proposition 5.7.5.

(i) *If* $q(K,o) = q(L,o)$ *and* $(K) \in \phi$ *then (up to conjugation)* $L \triangleleft K$ *and* K/L *is a torus*.

(ii) *Given* $L < G$ *there exists* $K \in \phi$ *such that* $L \triangleleft K$ *and* K/L *is a torus. Moreover we have in this case* $\varphi_L = \varphi_K$.

Proof.

(i) Since $q(K,o) = q(L,o)$ by 5.7.2 $\varphi_K = \varphi_L$. From $\chi(G/K^L) = \varphi_L(G/K) = \varphi_K(G/K) = |NK/K| \neq 0$, we see that G/K^L is non-empty and hence $(L) < (K)$. We take $L < K$. Let T be a maximal torus in NL/L and let P be its inverse image in NL. By 5.7.1 $q(P,o) = q(L,o)$. We show $(P) \in \phi$; then by 5.7.2 $(P) = (K)$. Assume $(P) \notin \phi$. Then NP/P contains a non-trivial maximal torus S. We let Q be its inverse image in NP. We claim that L is still normal in Q. Let $q \in Q$ induce the conjugation automorphism c_q on P. Since Q/P is a torus, c_q is homotopic to an inner automorphism, hence (e.g. by Conner-Floyd [47], 38.1) an inner automorphism itself and preserves the normal subgroup L. From the exact sequence

$$0 \longrightarrow P/L \longrightarrow Q/L \longrightarrow S \longrightarrow 0$$

and $P/L = T$ we conclude that Q/L is a torus and hence T is not a maximal torus.

(ii) Use the proof of (i) and 5.7.1.

As a corollary of 5.6.4 and 5.7.5 we obtain

Corollary 5.7.6.

Let $C(\Phi(G),Z)$ be the ring of continuous (= locally constant, in this case) functions. Then

$$(5.7.7) \qquad \varphi : A(G) \longrightarrow C(\Phi(G),Z)$$

$\varphi(x) : (H) \longmapsto \varphi_H(x)$, *is defined and an injective ring homomorphism.*

The possible equalities $q(H,p) = q(K,p)$ are not so easy to describe. We show that in a certain sense 5.7.1 is the only reason for such equalities. Given $K < G$. If NK/K is not finite or $|NK/K| \equiv 0 \mod p$ we find a subgroup $K \triangleleft P$ with $q(K,p) = q(P,p)$ as follows: Either by the procedure in the proof of 5.7.5 we let P be the inverse image in NK of a maximal torus in NK/K or we let P be the inverse image in NK of a Sylow p-group of NK/K. Then $(P) \in \Phi$ but it may happen that $|NP/P| \equiv 0$ mod p. In this case we can iterate the procedure. Either we arrive after a finite number of steps at a group Q with $|NQ/Q| \not\equiv 0 \mod p$, or we get a sequence

$$P_0 = K \triangleleft P_1 \triangleleft P_2 \triangleleft P_3 \triangleleft \dots$$

of groups with $q(P_i,p) = q(P_{i-1},p)$ and $|NP_i/P_i| \equiv 0 \mod p$ for $i \geq 1$. Let in this case Q be the closure in G of $\bigcup P_i$ (this is the limit in the space of subgroups, see 5.6). By continuity 5.6.4 we still have

$q(Q,p) = q(K,p)$. Now again we can apply our construction to Q if $|NQ/Q| \equiv 0 \bmod p$. Sooner or later we arrive at the defining group L of the prime ideal with $|NL/L| \not\equiv 0 \bmod p$.

That an infinite chain as above can actually occur is shown by the group $G = O(2)$. The groups in ϕ are $O(2)$, $SO(2)$ and the dihedral groups D_m. We have $ND_m = D_{2m}$. Hence

$$q(D_m,2) = q(D_n,2) \qquad \text{if} \qquad n = 2^j m.$$

For finite G the situation is more tractable.

Proposition 5.7.8.

Suppose $q(H,p) = q(K,p)$, $H \in \phi$, $K \in \phi$, $|NH/H| \not\equiv 0 \bmod p$, $|K/K_o| \not\equiv 0(p)$ where K_o is the component of the identity in K, and $p \neq 0$. Then up to conjugation $K \triangleleft H$ and H/K is a finite p-group.

Proof.

Choose P such that $NK > P > K$ and P/K a Sylow p-group of NK/K. We claim that $NP < NK$. Take $a \in NP$ and let K^a be the a-conjugate of K. Then $K/(K \cap K^a) < P/K^a$, hence $K/(K \cap K^a)$ is a finite p-group. On the other hand K, K^a, and P have the same component K_o of the identity, hence $K/(K \cap K^a)$ is a quotient of K/K_o which has order prime to p by assumption. Therefore $K = K \cap K^a = K^a$ and $a \in NK$. But then $|NP/P| \not\equiv 0 \bmod p$, because P/K was a Sylow p-group of NK/K. Now 5.7.1 and 5.7.2 imply $(P) = (H)$ and hence the assertion.

In particular if G is finite and $|NH/H| \not\equiv 0 \bmod p$ then there exists a unique smallest normal subgroup H_p of H such that H/H_p is a p-group and we have (with these notations)

Proposition 5.7.9.

$q(H,p) = q(K,p)$ *if and only if* $(H_p) \leq (K) \leq (H)$.

We shall see later that the cokernel of 5.7.7 is a torsion group of bounded exponent. We now make some remarks on the topology of Spec A(G), the prime ideal spectrum of A(G) with the Zariski topology.

Proposition 5.7.10.

The map

$$q : \phi(G) \times \text{Spec } Z \longrightarrow \text{Spec } A(G)$$

$$(H),\ (p) \longmapsto q(H,p)$$

is continuous, closed and surjective.

Proof.

An element $x \in C(\phi(G),Z) =: C$, being a locally constant function, is an integral linear combination of idempotent functions. Therefore this ring is integral over any subring. By an elementary result of commutative algebra (Atiyah-Mac Donald [11] , p. 67, Exercise 1) the mapping

$$\text{Spec } \varphi \ : \text{Spec } C \longrightarrow \text{Spec } A(G)$$

is closed (and surjective by 5.7.2). Hence the Proposition follows from the next Lemma.

Lemma 5.7.11.

Let X *be a compact, totally disconnected space. Then*

$$(x,(p)) \longmapsto \{f \mid f(x) \in (p)\}$$

<u>defines a homeomorphism</u>

$$F : X \times \text{Spex } Z \longrightarrow \text{Spec } C(X,Z).$$

<u>Proof.</u>

We ask the reader to recall the topology on Spec (Bourbaki [33], Ch. II). Certainly $\{f \mid f(x) \in (p)\}$ is a prime ideal in $C(X,Z)$ for any x and (p), so that F is defined. To define an inverse, let $k : Z \to C(X,Z)$ take n to the constant function $k_n : x \longmapsto n$. This induces a continuous map $k^* : \text{Spec } C(X,Z) \longrightarrow \text{Spec } Z$. Given $b \in \text{Spec } C(X,Z)$, let p be the element generating $k^* b$. Then we claim that $P = \bigcap_{f \in b} f^{-1}(p)$ consists of a single element of X. For if $p \neq 0$ and P is empty, then for each $x \in X$ there is a function $g_x \in b$ with $g_x(x) \notin (p)$. Since $k_p \in b$, for each $x \in X$ there is an $f_x \in b$ with $f_x(x) = 1$, i.e. the sets $f_x^{-1}(1)$ form a closed-open cover of X. Choose a finite subcover

$$U_i = f_{x_i}^{-1}(1), \qquad 1 \leq i \leq n.$$

Then one shows by induction on i that the characteristic function $K(V_i)$ of $V_i = U_i \cup \ldots \cup U_i$ is in b and in particular $k_1 \in b$, a contradiction. For $p = 0$, the same type of argument shows that k_m with $m = \text{l.c.m. } (g_{x_i}(x_i))$ is in b, contradicting $k^* b = (o)$. But if $x,y \in P$, choose $f \in b$ with $f(x) \in (p)$, and choose a closed-open U with $x \in U$, $y \in U$. Then setting

$$f_1 = f\, K(U) + (1-K(U))$$

$$f_2 = f(1-k(U)) + K(U)$$

we have $f_1 f_2 = f \in b$. Since $f_2(x) = 1$, $f_2 \notin b$, hence $f_1 \in b$, but $f_1(y) = 1$, hence $y \notin P$. Now we have a map $d : \text{Spec } C(X,Z) \longrightarrow X$ taking b to the unique element P, and the maps F and $d \times k^*$ are clearly inverse.

For the continuity of $d \times k^{*}$ we need only show d continuous. But for a closed -open $V \subset X$, $d^{-1}(V) = \{ b \mid K(V) \notin b \}$, which is open, while such V form a base of the topology of X.

It remains to be seen that F itself is continuous. But if $U = \{ b \mid f \notin b \}$ is a basic open set for some $f \in C(X,Z)$, and $q \in U$, then writing q as $F(x,(p))$ we have $f(x) = m \notin (p)$, and $V = f^{-1}(m)$ is closed-open in X containing x. Thus $q \in F(V \times \{(p) \mid m \notin (p)\}) \subset U$.

5.8. Relations between Euler-Characteristics.

We have described the Burnside ring of finite G-sets using congruences among fixed point sets (see 1.3). We generalize this description to compact Lie groups. The geometric interpretation of the Burnside ring then shows that we obtain a complete set of congruences that hold among the Euler characteristics of fixed point sets. We have already used the classical relations:

(5.8.1) $\chi(X) \equiv \chi(X^{P}) \bmod p$, P a p-group

(5.8.2) $\chi(X) = \chi(X^{T})$, T a torus.

Using 5.8.2 we have shown in 5.7 that is suffices to consider subgroups H with finite index in their normalizer. Therefore we pose the problem: Describe the image of

$$\varphi : A(G) \longrightarrow C(\phi(G),Z) =: C$$

The next Proposition shows that this can be done by using congruences.

Proposition 5.8.3.

C is a free abelian group with basis $x_H = |NH/H|^{-1} \varphi(G/H)$, $(H) \in O(G)$.

Proof.

A priori the x_H are only contained in $C \otimes Q$. But since NH/H acts freely on every fixed point set G/H^K, $(K) \in \Phi(G)$, we see that the numbers $\chi(G/H^K)$ are divisible by $|NH/H|$, and therefore $x_H \in C$. The elements x_H are linearly independent over Z because the G/H are. We have to show that each $x \in C$ is an integral linear combination of the x_H. Since x is continuous it attains only a finite number of values. Let $(H_1), \dots, (H_k)$ be the maximal elements of $\Phi(G)$ such that $x(H_i) \neq 0$. Consider $x - \sum_{1 \leq i \leq k} x(H_i) x_{H_i} =: y \in C$. If $y(K) \neq 0$ then (K) is strictly smaller than one of the (H_i). Induction, using the descending chain condition for subgroups, gives the result.

Now let X be a compact G-ENR. For $(H) \in \Phi(G)$ we consider the NH/H-space X^H. Since NH/H is a finite group we obtain as in 1.3

$$\sum_{n \in NH/H} \chi_{NH/H}(X^H)(n) \equiv 0 \mod |NH/H|$$

and this congruence can be rewritten in the form, using 5.3.,

$$(5.8.4) \qquad \sum_{(K)} n(H,K) \, \chi(X^K) \equiv 0 \mod |NH/H| ,$$

where the sum is taken over conjugacy classes (K) of $K < G$ such that $K \triangleright H$ and K/H is cyclic; the $n(H,K)$ are integers such that $n(H,H) = 1$.

Proposition 5.8.5.

The congruences 5.8.4 are a complete set of congruences for the image of $\varphi : A(G) \longrightarrow C$, i.e. $z \in C$ is contained in $\varphi A(G)$ if and only if for all $(H) \in \Phi(G)$

$$\Sigma_{(K)}\ n(H,K)\ z(H) \equiv 0 \bmod |NH/H|,$$

<u>with the summation convention as for</u> 5.8.4.

<u>Proof.</u>

Write z according to 5.8.3 as integral linear combination $z = \Sigma\, n_K\, x_K$ and suppose that z satisfies the congruences. If we can show that n_K is divisible by $|NK/K|$ then $z \in \varphi\, A(G)$. Choose (H) maximal with $n_{(H)} \neq 0$. Consider the congruence belonging to H. The only term which is non-zero is $n(H,H)\ z(H) = n_H$ which has to be zero mod $|NH/H|$. Therefore $n_H\ x_H \in \varphi\, A(G)$. Apply the same argument to $z - n_H\ x_H$ etc. Induction on the "length" of z in terms of the x_K gives that $z \in \varphi\, A(G)$.

Proposition 5.8.5 tells you which congruences hold among the Euler-Characteristics of fixed point sets X^H if X is a compact G-ENR. One would like to know the most general class of spaces for which such congruences hold. We must ensure that the results of 5.3 are applicable: The equivariant Euler-Characteristics $\chi_{NH/H}(X^H)$ should be defined and the decomposition formula 5.3 should hold.

<u>Remark 5.8.6.</u>

A different proof for 5.8.5 in the more general context of certain modules over A(G) was given in tom Dieck - Petrie [69] .

<u>Remark 5.8.7.</u>

As in 1.2.4 one shows that $\varphi : A(G) \longrightarrow C$ can be recovered from the ring structure of A(G): namely φ is the inclusion of A(G) into the integral closure in its total quotient ring.

5.9. Finiteness theorems.

We collect some finiteness theorems for compact Lie groups.

Proposition 5.9.1.

Let M be a compact differentiable G-manifold. Then M has only a finite number of orbit-types.

Proof.

Induct over dim M. An equivariant tubular neighbourhood U of an orbit $X \subset M$ is a G-vector bundle hence has only isotropy groups appearing on X or on the unit sphere of a fibre. By induction U has finite orbit type. (See Palais [124] , 1.7.25 for more details.)

Proposition 5.9.2.

Let G be a compact Lie group. There are only a finite number of conjugacy classes of subgroups which are normalizers of connected subgroups.

Proof.

(Bredon [26] , VII Lemma 3.2) Let L be the Lie algebra of G, E its exterior algebra, and $P(E)$ the projective space of E. If h is a linear subspace of L with basis $h_1,\dots,h_k$ then $h_1 \wedge \dots \wedge h_k$ determines a point ph of $P(E)$ which is independent of the choice of the basis. The adjoint action of G on L induces an action of G on $P(E)$. A subgroup N of G leaves h invariant if and only if ph is fixed under N. If H is a subgroup with Lie algebra h then:

$$gHg^{-1} = H \iff ad(g)h = h \iff g(ph) = ph .$$

Thus $NH = G_{ph}$. Now apply 5.9.1.

Proposition 5.9.3.

A compact Lie group G contains only a finite number of conjugacy classes (K) where K is the centralizer of a closed subgroup.

Proof.

Let G act on M = G via conjugation. If $H < G$ then M^H is the centralizer ZH. Apply 5.9.1.

We now come to a classical theorem of Jordan. Let $\sigma(G)$ be the set of finite subgroups of G.

Theorem 5.9.4.

There exists an integer j, depending only on the dimension and the number of components of G, with the following properties: For each $H \in \sigma(G)$ there exists an abelian normal subgroup A_H of H such that $|H/A_H| < j$. Moreover the A_H can be chosen such that $H < K$ implies $A_H < A_K$.

Proof.

(Boothby and Wang [24]. Wolf [169].) Given integers k and d there are only a finite number of groups G with $|G/G_o| = k$ and dim G = d, up to isomorphism (see 5.9.5). These groups can therefore be embedded into a fixed O(n). Hence it suffices to prove the theorem for G = O(n). A simply proof may be found for instance in Wolf [169], p. 100 - 103.

Theorem 5.9.5.

There exist only a finite number of non-isomorphic compact Lie groups of a given dimension and number of components.

Proof.

This depends on various classical results. We only describe the ingredients.

We begin with connected groups G. Then G is of the form

$$G = (T \times H)/D$$

where T is a torus, H is compact semi-simple, D is a finite central subgroup of T x H such that $D \cap T$ and $D \cap H$ are trivial (Hochschild [97] XIII Theorem 1.3). Therefore the projection of D into H is injective with image contained in the center ZH of H. This center ZH is finite by a theorem of Weyl (Helgason [96] , II. 6.9.). Hence given T and H there only a finite number of G's. By the classification theorem for semi-simple groups there are only a finite number of H's (Bourbaki [34]). This establishes the theorem for connected groups.

For the general case one has to study finite extensions

$$1 \longrightarrow G_o \longrightarrow G \longrightarrow E \longrightarrow 1$$

where G_o is connected and E is finite. By the general theory of group extensions and the finiteness of the cohomology of finite groups (Mac Lane [112] , IV) one sees that the following has to be proved: There are only a finite number of conjugacy classes of homomorphisms $E \longrightarrow Aut(G_o)/In(G_o)$ into the group of automorphisms modulo inner automorphisms. In case G_o is a torus the required finiteness follows from the Jordan-Zassenhaus theorem (Curtis-Reiner [48] , §79) and the general case is easily reduced to this case.

Theorem 5.9.6.

Let G be a compact connected Lie group. Then there exist only finitely many conjugacy classes of connected subgroups of maximal rank.

Proof.

Borel - de Siebenthal [23] .

We now consider solvable groups. A compact Lie group is called solvable if it is an extension of a torus by a finite solvable group. The derived group $G^{(1)}$ of G is the closure of the subgroup generated by commutators. We put inductively $G^{(n)} = (G^{(n-1)})^{(1)}$. A group H is called perfect if $H = H^{(1)}$. If $1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$ is an exact sequence of compact Lie groups, then B is solvable if and only if A and C' are solvable. A compact Lie group G is solvable if and only if there exists an integer n such that $G^{(n)} = \{1\}$. We list the following elementary facts.

Proposition 5.9.7.

a) Any subgroup H of G has a unique minimal normal subgroup H_a such that H/H_a is solvable.

b) For each H there exists an integer n such that $H^{(n)} = H_a$.

c) H_a is a perfect characteristic subgroup of H.

d) $H = H_a$ if and only if H is perfect.

e) $(H) = (K) \Rightarrow (H_a) = (K_a)$.

f) $K \triangleleft H$, H/K solvable $\Rightarrow K_a = H_a$.

Proof.

a): If $K \triangleleft H$, $L \triangleleft H$ and H/K, H/L are solvable then $K \cap L \triangleleft H$ and $H/K \cap L$ is solvable. By the descending chain condition for subgroups there is a minimal group as stated. b), c) and d): Since $H/H^{(1)}$ is abelian, by induction $H/H^{(k)}$ is solvable, hence $H^{(k)} > H_a$ for all k, and $H^{(k)}/H_a$ is solvable. If $H^{(k)} \neq H_a$ then $H^{(k)}$ has a non-trivial abelian quotient, hence $H^{(k)} \neq H^{(k+1)}$. By the descending chain condition there is an n such that $H^{(n)} = H^{(n+1)}$ and for this n necessarily $H^{(n)} = H_a$ and $H^{(n)}$ is perfect. The $H^{(n)}$ are characteristic subgroups.

e) and f) are obvious.

Theorem 5.9.8.

Let G be a compact Lie group. There exists an integer n such that for all $H < G$ we have $H^{(n)} = H_a$.

Proof.

Note that $H^{(n)} = H_a$ if and only if $(H/H_a)^{(n)}$ is the trivial group. Therefore we consider pairs H,K such that $H \trianglelefteq K < G$ and K/H is solvable. We show that there is an integer n such that for all such pairs $(K/H)^{(n)}$ is the trivial group. Let us call the smallest integer k such that $L^{(k)} = 1$ for a solvable group L the length $l(L)$ of L.

Take a pair K,H as above. Since K/H is solvable we have an exact sequence

$$1 \longrightarrow T \longrightarrow K/H \longrightarrow F \longrightarrow 1$$

where T is a torus and F is finite solvable. We have

$$l(K/H) \leq l(T) + l(T) = 1 + l(F).$$

So we need only show that the length of finite solvable subquotients is bounded. Let generally K_o denote the component of 1 of K. Then $K/H \longrightarrow F$ induces a surjection $p : K/K_o \longrightarrow F$. We show in a moment that there exists an integer $b(G)$ such that for any $K < G$ there exists an abelian normal subgroup A_K of K/K_o such that $|K/K_o : A_K| < b(G)$. Let be F_o be a pA_K. Then F/F_o has order less than $b(G)$. But $l(F) \leq l(F_o) + l(F/F_o) = 1 + l(F/F_o)$ because F_o is abelian. But $l(F/F_o)$ is bounded because only a finite number of groups occur.

The existence of the integer b(G) is proved by induction over dim G and $|G : G_o|$. Given G, the bound exists for the finite subgroups of G by Theorem 5.9.4. Let K be a subgroup of positive dimension. Consider

$$K_o < K < NK < NK_o.$$

Then K/K_o is a finite subgroup of $NK_o/K_o =: U$, and dim U < dim G. By 5.9.2. only a finite number of U occur up to isomorphism. This gives by induction the required finiteness.

We put WH = NH/H.

Theorem 5.9.9.

There exists an integer b such that for each closed subgroup H of G the index $|WH : (WH)_o|$ is less than b.

Proof.

The proof proceeds in three steps: We first reduce to the case that WH is finite; then we reduce to the case that H is finite; and finally we show that for finite H with finite WH the order of WH is uniformly bounded.

The group Aut H/In H of automorphism modulo inner automorphisms is discrete. Conjugation induces an injective homomorphism

$$NH/ZH \cdot H \longrightarrow \text{Aut } H/\text{In } H$$

where ZH is the centralizer of H. Hence NH/ZH·H being compact and discrete is finite. Hence

Lemma 5.9.10.

WH is finite if and only if $ZH/ZH \cap H$ is finite.

Lemma 5.9.11.

For any $H < G$ the group $ZH \cdot H$ has finite index in its normalizer.

This follows from the previous Lemma and the relations $Z(ZH \cdot H) < ZH < ZH \cdot H$.

If $n \in G$ normalizes H then also ZH and hence $ZH \cdot H$. We therefore have

$$NH/ZH \cdot H < N(ZH \cdot H)/ZH \cdot H \ .$$

Using Lemma 5.9.11 and the existence of an upper bound for the set

$$F(G) := \{ |WH| \mid H < G, \ WH \ \text{finite} \}$$

we obtain

Lemma 5.9.12.

There exists an integer c such that for all $H < G$ we have $|NH/ZH \cdot H| < c$.

Now we obtain the first reduction of our problem. From the exact sequence

$$1 \longrightarrow ZH/ZH \cap H \longrightarrow WH \longrightarrow NH/ZH \cdot H \longrightarrow 1$$

we see that $WH/(WH)_o \longrightarrow NH/ZH \cdot H$ has the kernel which is a quotient of $ZH/(ZH)_o$. Now Proposition 5.9.3 and Lemma 5.9.12 show that

$$\{ |WH/(WH)_o| \mid H < G \}$$

is bounded.

We show by induction over $|G/G_o|$ and dim G that F(G) has an upper bound $a = a(G/G_o, \dim G)$. For finite G we can take $a = |G|$. Suppose that an upper bound $a(K/K_o, \dim K)$ is given for all K with $\dim K < \dim G$. Let $T(G) = \{H < G \mid WH \text{ finite}\}$. Suppose $H \in T(G)$ is not finite. We consider the projection

$$p : NH_o \longrightarrow NH_o/H_o =: U \ .$$

Let V be the normalizer of H/H_o in U. Then $WH = V/(H/H_o)$ and therefore $H/H_o \in T(U)$. Since $\dim U < \dim G$ we obtain by induction hypothesis $|WH| \leq a(U/U_o, \dim U)$. We show that the possible values for $|U/U_o|$ are finite: This follows from 5.9.2. Hence for a given G the possible $|U/U_o|$ are bounded, say $|U/U_o| \leq m(G)$. We have

$$|U/U_o| \leq |G/G_o| \ m(G_o) \ .$$

By the classification theory of compact connected Lie groups there are only a finite number in each dimension. Hence there exists a bound for $|U/U_o|$ depending only on $|G/G_o|$ and dim G. This proves the induction step as far as the non-finite H in T(G) are concerned.

For the remaining case we use 5.9.4. and 5.9.6.

If $H \in T(G)$ is finite then also $K = NH$ is finite and by Lemma 5.9.11 $K \in T(G)$. We choose $j = j(|G/G_o|, \dim G)$ and A_H, A_K according to 5.9.4. We have

$$|K/H| \leq |K/A_K| \cdot |A_K/H \cap A_K| \leq j|A_K/H \cap A_K| .$$

Hence it suffices to find a bound for the $|A_K/H \cap A_K|$. Consider the exact sequence $1 \longrightarrow A_H \longrightarrow H \longrightarrow S \longrightarrow 1$. The conjugation c(a) with

$a \in A_K$ is trivial on A_H, because $A_K > A_H$, and hence $c(a)$ induces an automorphism of S. Since $|S| \leq j$ this automorphism has order at most $J = j!$, i. e. $c(a^r)$ is the identity on S and A_H for a suitable $r \leq J$. The group of such automorphisms modulo the subgroups of inner automorphisms by elements of A_H is isomorphic to $H^1(S;A_H)$, with S acting on A_H by conjugation. Since this group is annihilated by $|S|$ we see that $c(a^s)$ is an inner automorphism by an element of A_H for a suitable $s \leq J|S| \leq jJ$. In other words: $a^s h^{-1} \in ZH$. Hence it is sufficient to find a bound for the order of

$$A_K \cap ZH/H \cap A_K \cap ZH \ .$$

Let $U_1 = A_K \cap ZH$. By Borel-Serre [28] , Théorème 1, U_1 is contained in the normalizer NT of a maximal torus of G. Put $U = U_1 \cap T$. Then $|U_1/U| \leq |G/G_o||wG_o|$ where wG_o denotes the Weyl group of G_o. We estimate the order of U. Since U is abelian we have $U < ZU$. Moreover $H < ZU$ by definition of ZH. Since U is contained in the center $C = CZU$ of ZU. The inclusion $H < ZU$ implies $C < NH$. Hence C is finite.

We proceed to show that for the order of a finite center C(G) of G there exists a bound depending only on $|G/G_o|$ and dim G. We let G/G_o act by conjugation on $C(G_o)$ Then $C(G) \cap G_o$ is the fixed point set of this action. We have $C(G_o) = A \times T_1$, where A is a finite abelian group and T_1 is a torus. The group A is the center of a semisimple group and therefore $|A|$ is bounded by a constant c depending only on dim G. The exact cohomology sequence associated to the universal covering

$$0 \longrightarrow \pi_1 T_1 \longrightarrow V \longrightarrow T_1 \longrightarrow 0$$

shows, that the fixed point set of the action of G/G_o on $T_1 = C(G_o)_o$ is isomorphic to $H^1(G/G_o, \pi_1 T_1)$, hence its order is bounded by a constant d depending only on $|G/G_o|$ and the rank of T_1. Hence

$$|C(G)| \leq |G/G_o| \, cd \, .$$

Finally we show that for the possible groups ZU the order $|ZU/(ZU)_o|$ is bounded. U is contained in a maximal torus of G. Therefore ZU is a subgroup of maximal rank and $(ZU)_o$ a connected subgroup of maximal rank. By Theorem 5.9.6 there exist only finitely many conjugacy classes of connected subgroups of maximal rank. We have

$$|ZU/(ZU)_o| \leq |G/G_o| \, |N_o(ZU)_o/(ZU)_o| \, .$$

There are only finitely many possibilities for normalizers $N_o(ZU)_o$ in G_o of $(ZU)_o$.

This finishes the proof of Theorem 5.9.9. .

The last Theorem together with Proposition 5.8.3 gives the following result.

Proposition 5.9.13.

Let n *be the least common multiple of the numbers* $|NH/H|$ *where* $(H) \in \Phi(G)$. *Then the cokernel of* $A(G) \longrightarrow C(\Phi(G),Z)$ *is annihilated by* n.

5.10. Finite extensions of the torus.

We have seen earlier that the appearance of infinitely many elements in $\Phi(G)$ is connected with subgroups of G which are not semi-simple. The typical situation is given, when G itself is an extension of a torus T by a finite group F

$$a \longrightarrow T \longrightarrow G \xrightarrow{p} F \longrightarrow 1. \qquad (5.10.1)$$

In particular if we are given a homomorphism $h = F \longrightarrow \mathrm{Aut}(T) = GL(n,Z)$, $n = \dim T$, we can form the semi-direct product of T with F and h as twisting, call this G_h. Note that h is an integral representation of F. It would be interesting to know what the Burnside ring $A(G_h)$ can say about the integral representation (or vice versa). We are going to make a few elementary remarks concerning the Burnside ring A(G) for groups G as in 5.10.1.

Given G as in 5.10.1 let $h : F \longrightarrow \mathrm{Aut}\ (T)$ be the homomorphism induced by conjugation. We call a pair (F',T') with $F' < F$, $T' < T$ and T' invariant under F' _admissible_, and call $H < G$ an _(F',T')-subgroup_ if $p(H) = F'$ and $H \cap T = T'$.

Let $\sigma \in H^2(F,T)$ be the class given by 5.10.1. We have maps

$$k_* : H^2(F',T) \longrightarrow H^2(F',T/T')$$

$$i^* : H^2(F,T) \longrightarrow H^2(F',T) \quad .$$

Elementary diagram chasing then tells us

Proposition 5.10.2.

An (F',T')-subgroup exists in G if and only if $\sigma \in \mathrm{Ker}(k_* i^*)$.

Now choose any section $s : F \longrightarrow G$ to p and parametrize G as F x T via $g \longmapsto (pg, spg^{-1} \cdot g)$. The multiplication in G takes the form

$$(5.10.3) \qquad (f,t)(f',t') = (ff',,(t)f' + t' + \mu(f,f'))$$

where $(t)f = g^{-1}tg$ for $g \in p^{-1}(f)$ and

$$\mu(f,f') = s(ff')^{-1} s(f)s(f') .$$

We always assume $s(1) = 1$ from now on.

<u>Proposition 5.10.3.</u>

If H is an (F',T')-subgroup of G, and s is a section with $s(F') \subset H$, then a 1-1 correspondence between the (F',T')-subgroups of G and the crossed homomorphisms $\alpha : F' \longrightarrow T/T'$ is established by associating to H' the crossed homomorphism

$$\alpha(f') = k(s(f')^{-1}h(f'))$$

for h(f') any element of $H' \cap p^{-1}(f')$.

We leave the proof as an exercise. We denote the group described in 5.10.3 by (F',T',α). If G is a semi-direct product then $\sigma = 0$ and (F',T')-subgroups always exist; in this case it is advisable to choose s as a homomorphism.

We now describe the effect of conjugation. For conjugation by elements of T, note that in our parametrization

$$(1,t)(f',t')(1,t)^{-1} = (f',t' + (t)f' - t) .$$

Thus denoting by $d_t : F' \longrightarrow T$ the principal crossed homomorphism

$d_t(f') = (t)f'-t$, the result of conjugating (F',T',α) by $(1,t)$ is (F',T',α') with $\alpha'(f') = \alpha(f') + k(d_t(f'))$.

Proposition 5.10.4.

Given a choice of H and s as in 5.10.3. There is a 1-1 correspondence between classes of (F',T')-subgroups under conjugation by elements of T and the elements of $H^1(F',T/T')$.

Proposition 5.10.5.

If $H < G$ *is an (F',T')-subgroup then*

$$NH \cap T/H \cap T = \mathrm{Fix}(F',T/T') .$$

Proofs are again left as exercises.

Proposition 5.10.6.

If H is an (F',T')-subgroup then the following are equivalent:

i) $H \in \Phi(G)$.

ii) $\mathrm{Fix}(F',T/T')$ *is finite*

iii) T' *contains the zero-component of* $\mathrm{Fix}(F',T)$.

Proof.

The equivalence i) $\Leftrightarrow$ ii) follows immediately from 5.10.5. The equivalence i) $\Leftrightarrow$ iii) is elementary Lie group theory and will be left to the reader.

From 5.10.2. and 5.10.6 one obtains

Proposition 5.10.7.

$\Phi(G)$ *is infinite if and only if the action of F on T is non-trivial.*

This can be used to give an analogous result for an arbitrary compact Lie group.

Proposition 5.10.8.

Let G be a compact Lie group. Then $\Phi(G)$ is finite if and only if the action of the Weyl group on the maximal torus is trivial.

Proof.

If the action is trivial then G_o can have no semi-simple component. Hence G is of type 5.10.1 and 5.10.6 says that $\Phi(G)$ is finite.

Now assume that in

$$0 \longrightarrow T \longrightarrow NT \longrightarrow WT \longrightarrow 1 ,$$

T a maximal torus, the action of the Weyl group WT on T is non-trivial. By 5.10.7 $\Phi(NT)$ is infinite. We show that an infinite number of elements of $\Phi(NT)$ are contained in $\Phi(G)$. We know that $NT = \lim H_i$, $H_i \neq NT$. By continuity our assertion follows with the help of the next Lemma.

Lemma 5.10.9.

Let $H < K < G$. Then $(H) \in \Phi(G)$ if and only if $(H) \in \Phi(K)$ and G/K^H is finite.

Proof.

If $(H) \in \Phi(G)$ then, of course, $(H) \in \Phi(K)$ and G/K^H is finite because it consists of a finite number of NH/H-orbits. For the other direction, note that $H < N_KH < N_GH$ yields a fibre bundle $N_KH/H \rightarrow N_GH/H \longrightarrow N_GH/N_KH$. But the inclusion $N_GH \longrightarrow G$ induces an injective map

$$N_GH/N_KH = N_GH/N_GH \cap K \longrightarrow G/K^H .$$

Thus if $(H) \in \Phi(K)$ and G/K^H is finite, both base and fibre are finite.

We now report briefly about cyclic extensions of a torus (see Gordon [87]).

Proposition 5.10.10.

If G is an extension of T by F and F' < F is cyclic, then any two (F',T') subgroups of G are conjugate under an element of T.

Proof.

If F' is cyclic and Fix(F',T/T') is finite, then $H^1(F',T/T') = 0$. Now use 5.10.4.

If f is cyclic of order n with generator f and M is any F-module then

$$H^2(F,M) \cong Fix(F,M)/N^*M$$

where $N^*M = \sum_{i=o}^{n-1} (m)f^i$. Since for an r-torus $M = T^r$ we have $H^2(F,T^r) \cong H^3(F,Z^r)$ this group is finite. Thus N^*T^r contains the zero-component of $Fix(F,T^r)$. On the other hand, if $\varphi : I \longrightarrow T^r$ is any path from 0 to t, then $\sum_{i=o}^{n-1} (\varphi)f^i$ is a path in N^*T^r from 0 to $\sum_{i=o}^{n-1} (t)f^i$, so that N^*T^r is connected. Hence for any torus T^r, N^*T^r is precisely $Fix(F,T^r)_o$.

The isomorphism $H^2(F,T) \cong Fix(F,T)/N^*T$ means that the extension G is characterized by a component of Fix(F,T). Now note that it is no essential restriction to assume $N^*T = 0$. For if L is any compact Lie group and $(ZL)_o$ the zero-component of its center, then $L \longrightarrow L/(ZL)_o$ induces an isomorphism of rings

$$A(L/(ZL)_o) \cong A(L) .$$

Now choose any element $s(f) \in p^{-1}(f)$ and construct a section s by putting $s(f^i) = s(f^i)$, $0 \leq i < n$. Then since $s(f)^{-1}s(f)^n s(f) = s(f)^n$, $\tau = s(f)^n \in Fix(F,T)$, and τ is the image of $[G] \in H^2(F,T)$ in $Fix(F,T) = Fix(F,T)/N^*T$. If now (F',T') is an admissible pair then there exists an (F',T') subgroup H with NH/H finite if and only if both the zero-component and the τ-component of $Fix(F',T')$ are in T'.

Suppose $\tau \in Fix(F,T)$, latter being discrete, and let T' be the (finite) subgroup generated by τ. Then T/T' inherits an F-operation. With these notations one has

<u>Theorem 5.10.11.</u>

<u>*If* G *is the extension of* T *by* F *defined by* τ, *and* G' *is the semi-direct product of* T/T' *and* F *in the action above, then* $A(G) \cong A(G')$.</u>

<u>Proof.</u>

There exists a map $t : G \longrightarrow G'$ making the following diagram commutative

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & F \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle id} \\ T/T' & \longrightarrow & G' & \longrightarrow & F \end{array} \quad .$$

By the analysis of (F',T')-subgroups of G given above it is seen that t induces the required isomorphism.

5.11. Idempotent elements.

In section 1.4 we have described the idempotents of A(G) for finite G. We generalize this to compact Lie group, using results of 5.9. and 5.6.

Let S = S(G) be the space of closed subgroups of G and cS the quotient space under the conjugation action (see 5.6). Let $H^{(1)}$ be the commutator subgroup of H and H_a the smallest normal subgroup of H such that H/H_a is solvable (see 5.9.8). Let P be the space of perfect subgroups in S

Proposition 5.11.1.

The maps $H \longmapsto H^{(1)}$ *and* $H \longmapsto H_a$ *are continuous maps* $S \longrightarrow S$. *The space* P *is closed in* S.

Proof.

In view of the compactness of S and 5.9.8 we need only show that $H \longmapsto H^{(1)}$ is continuous. Let $H_1, H_2, \ldots$ be a sequence of subgroups converging to H. Without loss of genrality we can assume that the H_i are conjugate to subgroups of H. Moreover by 5.6.2 we can find a sequence $g_i \in G$ converging to 1 such that $K_i = g_i H_i g_i^{-1}$ is contained in H. We show that $\lim K_i^{(1)}$ exists and is equal to $H^{(1)}$. Fix $\varepsilon > 0$ and choose n such that in the Hausdorff metric $d(K_i, H) < \varepsilon$ for $i \geq n$. Let $c^k K$ be the closed subspace of a group K consisting of elements which are product of a most k commutators. Then $d(K_i, H) < \varepsilon$ implies $d(c^k K_i, c^k H) < 4k\varepsilon$ Choose k such that $d(c^k H, H^{(1)}) < \varepsilon$. Then for $i \geq n$ we have $d(c^k K_i, H^{(1)}) < (4k+1)\varepsilon$ and a fortiori $d(K_i^{(1)}, H^{(1)}) < (4k+1)\varepsilon$.

As a corollary we obtain

Proposition 5.11.2.

Given a perfect subgroup H *of* G. *Then* $\{K \mid K_a = H\}$ *and* $\{K \mid K_a \sim H\}$ *are closed subsets of* S.

In 5.7 we obtained the closed quotient map

$$q : S \times \mathrm{Spec}\, Z \longrightarrow \mathrm{Spec}A(G) : (H),(p) \longmapsto q(H,p) .$$

Let r be the composition

$$S \times \mathrm{Spec}\, Z \xrightarrow[pr]{} S \xrightarrow[a]{} P \xrightarrow[c]{} cP$$

where pr is the projection, a the map $aH = H_a$, and c the map $cH = (H)$ into the space cP of conjugacy classes of perfect subgroups. Then r is continuous by 5.11.1.

Proposition 5.11.3.

The map r *factors over* q *inducing a continuous surjective map* $s : \mathrm{Spec}\, A(G) \longrightarrow cP$.

Proof.

Suppose $q(H,p_1) = q(K,p_2)$. Since p is the residue characteristic of $q(H,p)$ we must have $p_1 = p_2$. Put $p = p_1$. Let (H^*) be the unique conjugacy class such that $q(H,p) = q(H^*,p)$ and NH^*/H^* is finite (see 5.7.2). By 5.7 we can find a countable transfinite sequence $H \triangleleft H_1 \triangleleft H_2 \dots H_\lambda \sim H^*$ such that H_{i+1}/H_i is solvable and H_j is the limit of the preceeding subgroups if j is a limit ordinal. It follows from Proposition 5.11.1 that $H_a = (H_\lambda)_a$.

The space cP being a countable compact metric space is totally disconnected. Hence we get a unique continuous map e which makes the following diagram commutative

$$\begin{array}{ccc} & \text{Spec } A(G) & \\ {}^{s}\swarrow & & \searrow^{\pi} \\ cP & \xrightarrow[e]{\qquad\qquad} & \pi \text{ Spec } A(G) \end{array}$$

Here π is the projection onto the space of components.

<u>Theorem 5.11.4.</u>

<u>The map</u> e <u>is a homomorphism</u>.

<u>Proof.</u>

π Spec A(G) is a quotient of a quasi compact space hence quasi compact. The space cP is a Hausdorff space. We therefore need only show that e is bijective. We already know that e is surjective.

Given two components B and C of Spec A(G). Choose elements $q(H,p) \in B$, $q(K,1) \in C$. Assume that $e(B) = e(C)$, hence

$$(H_a) = sq(H,p) = sq(K,1) = (K_a).$$

Since H/H_a is solvable we can find a finite chain of subgroups

$$H = H_1 \triangleright H_2 \triangleright \dots \triangleright H_k = H_a$$

such that H_i/H_{i+1} is a torus or finite cyclic of prime order. By 5.7.1 $q(H_i,p_i) = q(H_{i+1},p_i)$ for a suitable prime. If $\bar{q}(H,p)$ denotes the closure of the point $q(H,p)$ we have

$$q(H,p) \in \bar{q}(H_1,o),\ \bar{q}(H_i,o) \cap \bar{q}(H_{i+1},o) \neq \emptyset ,$$

and therefore $q(H_a,o) \in B$. Similarly $q(K_a,o) \in C$ and therefore $B = C$.

We now show how Theorem 5.11.4 leads to a description of idempotent elements.

Let U be an open and closed subset of Spec A(G). Then U is a union of components and projects into an open and closed subset of cP called s(U). Let e(U) be the idempotent element of A(G) which corresponds to U (Bourbaki [33] , II. 4.3, Proposition 15). Let $S(U) = \{H < G \mid \varphi_H\ e=1\}$

Proposition 5.11.5.

$H \in S(U) \iff (H_a) \in S(U)$.

Proof.

Since e(U) is idempotent $\varphi_H(e(U))$ is 0 or 1. We have to recall how to pass from U to e(U). Let Z be the complement of U in Spec A(G). Then

$$Z = V(A(G)e(U)) = \{q \in \text{Spec } A(G) \mid q \supset A(G)e(U)\} .$$

Moreover $e(Z) = 1-e(U)$. Suppose $\varphi_H e(U) = 1$, then $\varphi_H(e(Z)) = 0$, so $\varphi_H\ A(G)e(Z) = (0)$, which means $q(H,o) \supset A(G)e(Z)$, $q(H,o) \in V(A(G)e(Z)) = U$ and therefore $(H_a) \in s(U)$.

Conversely, if $(H_a) \in s(U)$, then $q(H,o) \in U$, $\varphi_H\ A(G)e(Z) = (0)$, $\varphi_H\ e(U) = 1$.

The idempotent is indecomposable if and only if U is a component. If the perfect subgroup H of G is not a limit of perfect subgroups then

$\{q(K,p) \mid (K_a) = (H)\} := U(H)$ is a component and H yields an indecomposable idempotent $e_H := e(U(H))$.

We are now going to show that the topological considerations above are necessary in that usually an infinite number of conjugacy classes of perfect subgroups exists. Let $1 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 1$ be an exact sequence where T is a torus and F a finite group. Conjugation in G induces a homomorphism $\varphi : F \longrightarrow \mathrm{Aut}(T)$ which we also interprete as action of F on T (compare section 5.10.) Let F_U be the kernel of φ.

Proposition 5.11.6.

Let G be a finite extension of a torus as above. Then the number of conjugacy classes of perfect subgroups of G is finite if and only if F/F_U is solvable. If F/F_U is solvable then the set of perfect subgroups is finite.

Proof.

A quotient of a perfect group is perfect. Let F/F_u be solvable. Let $H < G$ be perfect. Then the image under $G \longrightarrow F \longrightarrow F/F_u$ is perfect hence trivial. Therefore H is an extension $1 \longrightarrow H \cap T \longrightarrow H \longrightarrow P \longrightarrow 1$ with $P < F_u$ perfect and trivial action of P on $H \cap T$ and T. Let K be the preimage of P under $p : G \longrightarrow F$. Then $H \triangleleft K$ since T is contained in the center of K. The group $K/H = T/H \cap T$ is solvable. Hence $H = K_a$. There a perfect group comes via the map $K \longmapsto K_a$ from a finite set of subgroups.

Now let us assume that F/F_u is not solvable. Let $P < F/F_u$ be a nontrivial perfect subgroup. Let H be the pre-image of P under $G \longrightarrow F/F_u$ and $Q < F$ be its group of components. Let T_o be the component of 1 in the fixed point set of the Q action on T. Since $Q > F_u$, $Q \neq F_u$, we have $T_o \neq T$. The group T_o is contained in the center of H and $H \longrightarrow H/T_o$ induces an injective ring homomorphism $A(H/T_o) \longrightarrow A(H)$. If $A(H/T_o)$ has an infinite number of idempotents then A(H) has an infinite number of idempotents, hence an infinite number of conjugacy classes of perfect subgroups. The action of Q on H/T_o has zero-dimensional fixed point set. Hence we have reduced the problem to the case $T_o = \{1\}$. But then a subgroup L of H which projects onto P under $H \longrightarrow Q \longrightarrow P$ has finite index in its normalizer. Let L be such a group and consider its derived group $L^{(1)}$. Then $L^{(1)}$ also projects onto P because P is perfect. Therefore $NL^{(1)}/L^{(1)}$ is finite and $L/L^{(1)} < NL^{(1)}/L^{(1)}$. But we have shown in 5.9.4 that there exists a number b such that for any $L < H$ with finite index in its normalizer $|NL/L| < b$. Together with 5.9.8 we see that there is an integer n such that L/L_a is finite of order less than b^n. Hence if there exists an infinite set of subgroups of H which projects onto P and which contains groups of arbitrary large order then the set of conjugacy classes of perfect subgroups is infinite. But infinite sequence of subgroups of the required sort is easily constructed, using the techniques of 5.10.

5.12. Functorial properties.

If X is a G-space and $H < G$ then X can be considered as H-space. This induces the forgetful functor r_H^G : G-Top $\longrightarrow$ H-Top from the category of G-spaces to the category of H-spaces. This functor has a left adjoint, called extension from H-spaces to G-spaces. On objects it is defined by

$$e_H^G(X) = Gx_H X$$

for an H-space X. The adjointness means that for H-spaces X and G-spaces Y we have a natural bijection

$$\mathrm{Map}_G(Gx_H X, Y) \cong \mathrm{Map}_H(X, r_H^G Y) \quad ,$$

where Map_G is the set of G-maps. If $f : X \longrightarrow Y$ is an H-map then $f' : Gx_H X \longrightarrow Y : (g,x) \longmapsto gf(x)$ is the adjoint G-map.

Proposition 5.12.1.

The assignment $X \longmapsto Gx_H X$ *induces an additive homomorphism*

$$e_H^G : A(H) \longrightarrow A(G).$$

(X *a compact* H-ENR.)

Proof.

Given $K < G$, then $Gx_H X)^K \neq \emptyset \Rightarrow G/H^K \neq \emptyset \Rightarrow (K) < (H)$. Assume $K < H$. We have to show that $\chi((Gx_H X)^K)$ can be computed from Euler-Characteristics of fixed point sets X^L. The set G/H^K is finite (if $K \in \Phi(G)$). The fibre of $(Gx_H X)^K \longrightarrow G/H^K$ over gH is homeomorphic to $X^{gKg^{-1} \cap H}$. Hence

$$\chi((Gx_H X)^K) = \sum_{gH \in G/H^K} \chi(X^{gKg^{-1} \cap H}) \quad .$$

If $f : H \longrightarrow K$ is a continuous homomorphism between compact Lie groups then a K-space X can be considered via f as an H-space. This induces a ring homomorphism

$$A(f) = f^{*} : A(K) \longrightarrow A(H)$$

and A(-) becomes a contravariant functor from compact Lie groups to commutative rings. If $f : H \subset K$ then f is called restriction, also denoted r_H^K.

We want to investigate the various interrelations between the e_H^G and r_H^G. We need a slightly more general map then the e_H^G. This is done best by redefining e_H^G and r_H^G using a more general concept than the Burnside ring.

Let S be a closed differentiable G-manifold and let a(S) be the set of differentiable G-maps $M \longrightarrow S$ which are proper submersions. On a (S) we induce the following equivalence relation : $p : M \longrightarrow S$ equivalent to $q : N \longrightarrow S$ if and only if for all $s \in S$ and all $H < G_s$ the equality

$$\chi(p^{-1}(s)^H) = \chi(q^{-1}(s)^H)$$

holds. Disjoint union (addition) and fibre product over S (multiplication) makes the set of equivalence classes into a commutative ring with identity, denotes A [S] . If S is a point this is the Burnside ring; hence we call A [S] the Burnside ring of G-manifolds over S. We are going to describe the functorial properties of this ring.

Let $f : T \longrightarrow S$ be a differentiable G-map. Let $p : M \longrightarrow S$ be a submersion as above. Then in the pull-back diagram

$$\begin{array}{ccc} N & \xrightarrow{F} & M \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

the map q is a proper submersion and defines an element in $A[T]$. The assignment $p \longmapsto q$ induces a ring homomorphism $f^* : A[S] \longrightarrow A[T]$.

We also have covariant maps. Let $f : T \longrightarrow S$ be a submersion. Then composition with f induces an additive (but not multiplicative) map $f_* : A[T] \longrightarrow A[S]$. These maps have the following properties.

Proposition 5.12.2.

i) f^* is a homomorphism of rings. We have $(id)^* = id$ and $(fg)^* = g^*f^*$.

ii) For any submersion $f : T \longrightarrow S$ the map f_* is well-defined and additive. We have $(id)_* = id$ and $(fg)_* = f_*g_*$.

iii) For $a \in A[S]$ and $b \in A[T]$ we have

$$af_*(b) = f_*(f^*(a)b).$$

iv) Let

$$\begin{array}{ccc} T' & \xrightarrow{F} & S' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

be a pull-back diagram with f and hence F a submersion. Then $p^*f_* = F_*P^*$.

v) If $f_o, f_1 : T \longrightarrow S$ are G-homotopic then $f_o^* = f_1^*$.

The proofs are straightforward and left to the reader. The connection with material at the beginning of this section if obtained using a canonical isomorphism $A[G/H] \cong A(H) : p : M \longrightarrow G/H \longmapsto p^{-1}(H/H)$.

Proposition 5.12.2 iv) generalizes the main property of Mackey functors in the sense of Dress [80] to compact Lie groups. But in the case of non-finite Lie groups there exists a double coset formula which is a less formal generalization of the Mackey axiom and is more accessible to computation. We are going to describe this formula.

We consider a pull-back diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \bar{h}\ } & G/L \\ {\scriptstyle \bar{k}}\downarrow & & \downarrow{\scriptstyle k} \\ G/K & \xrightarrow[\ h\ ]{} & G/P \end{array} \quad .$$

The problem is to compute $k^* h_*$. We use a decomposition of S into homogeneous spaces but slightly more refined than the decomposition in the Burnside ring. As in Section 5.5 we have the decomposition $S = \bigcup S_{(H)}$ into the subspaces of a given orbit type. We let $S_{(H),b}$ be the invers image in $S_{(H)}$ of the connected components of $S_{(H)}/G$. So the index b distinguishes the components. Then we still have a decomposition

$$S = \sum n_{(H),b} \left[M_{(H),b}\right]$$

in A(G) with $n_{(H),b} := \chi_c(S_{(H),b}/G) \in Z$ and $M_{(H),b}$ an orbit in $S_{(H),b}$. We let $k_{(H),b} : M_{(H),b} \longrightarrow G/K$ and $h_{(H),b}$ G/L be the maps which are compositions of the inclusion $M_{(H),b} \subset S$ with the maps $\bar{k}$ and $\bar{h}$ respectively. Then we claim

Theorem 5.12.3.

We have the equality of maps

$$k^* h_* = \sum_{(H),b} n_{(H),b} (h_{(H),b})_* (k_{(H),b})^* .$$

Proof.

Given an element x in A[G/K] represented by $f : M \longrightarrow G/K$. Then $k^* h_* x$ is represented by $\bar{h}F$ in the pull-back diagram below (where the squares and hence also the rectangle are pull-backs).

$$\begin{array}{ccccc}
\bar{M} & \xrightarrow{F} & S & \xrightarrow{h} & G/L \\
\downarrow & & \downarrow \bar{k} & & \downarrow k \\
M & \xrightarrow[f]{} & G/K & \xrightarrow[h]{} & G/P
\end{array} .$$

Since pull-backs are transitive the pull-back of $f : M \longrightarrow G/K$ along $k_{(H),b}$ is the fibre of $F : \bar{M} \longrightarrow S$ over $M_{(H),b}$, say $F_{(H),b} : \bar{M}_{(H),b} \rightarrow M_{(H),b}$ and this represents $k^*_{(H),b}x$. Hence $(h_{(H),b})_* (k_{(H),b})^* x$ is represented by the composition

$$h_{(H),b} F_{(H),b} : \bar{M}_{(H),b} \longrightarrow M_{(H),b} \longrightarrow G/L.$$

So we have to show that the following two elements are equal in A[G/L], namely $[\bar{h}F]$ and $\sum n_{(H),b} [h_{(H),b} F_{(H),b}]$. This means by definition of A[G/L] that we have to show: For each $U < L$ the U-fixed points of the fibres over the coset L/L of G/L have the same Euler-characteristic.

The fibre of $\bar{h}F$ is the fibre of hf over k(L/L), considered as

L-manifold. Since we are now dealing with G-spaces over G/L the whole situation can be reconstructed from the fibres over L/L, which we denote by an index zero, using canonical G-diffeomorphisms like $G \times_L \overline{M}^o = \overline{M}$. We have for $V < L$

$$M_{(H),b} = G \times_L M^o_{(H),b}, \quad S_{(V)} = G \times_L S^o_{(V)}, \quad S_{(V),b} = G \times_L S^o_{(V),b}$$

using the identification $S_{(V)}/G = S^o_{(V)}/L$.

Let $F : \overline{M}^o \longrightarrow S^o$ be the restriction of the map $F : \overline{M} \longrightarrow S$. As in Section 5.5 we have

$$(5.12.4) \qquad \chi((\overline{M}^o)^U) = \sum_{V,b} \chi_c((F^{-1}S^o_{(V),b})^U) .$$

The map

$$F^{-1}(S^o_{(V),b}) \longrightarrow S^o_{(V),b} \longrightarrow S^o_{(V),b}/L$$

is a fibre bundle with the fibre $F^{-1}(M^o_{(V),b})$ such that the U-fixed points again yield a fibration with typical fibre $F^{-1}(M^o_{(V),b})^U$. Then the ((V),b)-summand in (5.12.4) is by Proposition equal to

$$\chi_c(F^{-1}(M^o_{(V),b})^U) \ \chi_c(S^o_{(V),b}/L)$$

$$= \chi_c(F^{-1}(M^o_{(V),b})^U) \ \chi_c(S_{(V),b}/G)$$

$$= \chi_c(F^{-1}(M^o_{(V),b})^U) n_{(V),b}$$

and this was to be shown.

5.13. Multiplicative induction and symmetric powers.

Let K be a subgroup of finite index in G. Let $Hom_K(G,X)$, for a K-space X, be the space of K-maps $G \longrightarrow X$ with G-action induced by right translation on G'. The functor $X \longmapsto Hom_K(G,X)$ from K-spaces to G-spaces is right adjoint to the restriction functor and preserves in particular products. Explicitely, we have a natural bijection

$$Map_G(Y,Hom_K(G,X)) \cong Map_K(Y,X),$$

where Y is any G-space. Given $f : Y \longrightarrow Hom_K(G,X)$ in the set on the left we compose with the K-map $Hom_K(G,X) \longrightarrow X : f \longmapsto f(1)$ to obtain the corresponding element in the set on the right side. We have chosen K to be of finite index in G in order to avoid some technical problems: In our case $Hom_K(G,X)$ as a topological space is simply the product $\prod_{y\in G/K} X$ of $|G/K|$ copies of X.

Proposition 5.13.1.

<u>The assignment</u> $X \longmapsto Hom_K(G,X)$ <u>induces a map</u> $A(K) \longrightarrow A(G)$ <u>which, in general, is not additive but preserves products</u> (X a compact G-ENR).

Proof.

Given $H < G$ we have to compute $\chi(Hom_K(G,X)^H)$. Since K has finite index in G the space G/H is K-homeomorphic to a finite disjoint union $\coprod_i K/K(i)$ of homogeneous spaces. The equalities

$$\begin{aligned} Hom_K(G,X)^H &= Hom_G(G/H, Hom_K(G,X)) \\ &= Hom_K(G/H,X) \\ &= Hom_K(\textstyle\coprod_i K/K(i),X) \\ &= \textstyle\prod_i Hom_K(K/K(i),X) \\ &= \textstyle\prod_i X^{K(i)} \end{aligned}$$

show that the Euler-Characteristics in question can be computed from Euler-Characteristics of fixed point sets X^L, $L < K$.

We call $X \longmapsto Hom_K(G,X)$ and the map induced on the Burnside ring <u>multiplicative induction</u>.

<u>Proposition 5.13.2.</u>

<u>Let</u> L <u>be a finite normal subgroup of</u> G. <u>The assignment</u> $X \longmapsto X/L$ <u>induces a map</u> $A(G) \longrightarrow A(G/L)$. (X <u>a compact</u> G-ENR.)

<u>Proof.</u>

Given $H < G/L$ we have to show that $\chi(X/L^H)$ is determined by Euler-Characteristics of fixed point sets of X. Let P be the inverse image of H in G. Let $B = p^{-1}(X/L^H)$ where $p : X \longrightarrow X/L$ is the quotient map. We consider X and B as P-spaces. An Orbit of X isomorphic to P/U is contained in B if and only if P = LU. Hence B is a union of orbit bundles. From Proposition $[B] = [B']$ in A(P) where $B' \subset X'$ has a similar meaning as B. Now

$$\chi(X/L^H) = \chi(B/L) = |L|^{-1} \sum_{g \in L} \chi(B^g).$$

Here we have used 5.3.12. Hence $\chi(X/L^H)$ can be computed from Euler-Characteristics as we wanted. We still have to show that X/L is a G/L-ENR. By 5.2.6 if suffices to see that all $M/L^H = B/L$ are ENR. But B is an ENR by 5.2.6 and hence B/L an ENR by 5.2.5.

We now discuss symmetric powers. Let S_r be the symmetric group on r symbols. If X is a G-space then the diagonal action of G on X^r and the permutation action of S_r commute, so we can view X^r as $(S_r \times G)$-space. If M is an S_r-space with trivial G-action then $M \times X^r$ is an $(S_r \times G)$-space. Dividing out the S_r-action yields the G-space $(M \times X^r)/S_r$.

Proposition 5.13.3.

The assignment $(M,X) \longmapsto (M \times X^r)/S_r$ induces a map

$$A(S_r) \times A(G) \longrightarrow A(G).$$

(M,X compact G-ENR's.)

Proof.

We begin by showing that $X \longmapsto X^r$ induces a map $w : A(G) \longrightarrow A(S_r \times G)$. The standard embedding $S_{r-1} \subset S_r$ gives $S_{r-1} \times G$ as a subgroup of finite index in $S_r \times G$. Viewing X as an $(S_{r-1} \times G)$-space via the projection $S_{r-1} \times G \longrightarrow G$ then the $(S_r \times G)$-space X^r is obtained from X using the multiplicative induction corresponding to $S_{r-1} \times G < S_r \times G$. Therefore w is well-defined by 5.13.1. Now consider the following composition of maps

$$A(S_r) \times A(G) \xrightarrow{p \times w} A(S_r \times G) \times A(S_r \times G)$$
$$\xrightarrow{m} A(S_r \times G) \xrightarrow{q} A(G)$$

where w is as above, p is induced by the projection $S_r \times G \longrightarrow S_r$, m is ring-multiplication, and q is the quotient map of 5.13.2. We check that on representatives the above composition is $(M,X) \longmapsto (M \times X^r)/S_r$.

Let $\pi < S_r$ be a subgroup. Then X^r/π is the π-symmetric power, a G-space if X is a G-space. Note that $(S_r/\pi \times X^r)/S_r = X^r/\pi$. Hence we have

Corollary 5.13.4.

$X \longmapsto X^r/\pi$ induces a map $A(G) \longrightarrow A(G)$.

We are going to analyse the formal properties of the map 5.13.3. We write this map

(5.13.5) $$A(S_r) \times A(G) \longrightarrow A(G) : (x,y) \longmapsto x\cdot y .$$

We recall some constructions with the symmetric group. Let X,Y be S_r-, S_t-spaces, respectively. We write

(5.13.6) $$X\cdot Y = S_{r+t} \times_{S_r \times S_t} (X \times Y)$$

using the standard embedding $S_r \times S_t \subset S_{r+t}$.

Let $S_r \int G$ be the <u>wreath-product</u> of G with S_r. This is the set $S_r \times G^r$ with group-law

$$(s;g_1,\ldots,g_r)(t;h_1,\ldots,h_r) = (st;g_1 h_{s^{-1}(1)},\ldots,g_r h_{s^{-1}(r)})$$

If M is a G-space then M^r becomes an $S_r \int G$-space with action

$$(s;g_1,\ldots,g_r)(m_1,\ldots,m_r) = (g_1 m_{s^{-1}(1)},\ldots,g_r m_{s^{-1}(r)}) .$$

We consider $S_r \int S_t$ as a subgroup of S_{rt}: If $M = S_t$ as S_t-space then $S_r \int S_t$ acts as a group of permutations on M^r; now identify M^r in a sensible way with $\{1,2,\ldots,rt\}$. (The conjugacy class of $S_r \int S_t$ in S_{rt} is then uniquely determined.) Let X,Y be S_r-, S_t-spaces respectively. We write

(5.13.7) $$X * Y = S_{rt} \times_{(S_r \int S_t)} (X \times Y^r).$$

<u>Proposition 5.13.8.</u>

<u>The constructions</u> $(X,Y) \longmapsto X\cdot Y$ <u>and</u> $(X,Y) \longmapsto X * Y$ <u>induce maps</u>

$$A(S_r) \times A(S_t) \longrightarrow A(S_{r+t}) \ : \ (x,y) \longmapsto x \cdot y$$

$$A(S_r) \times A(S_t) \longrightarrow A(S_{rt}) \ : \ (x,y) \longmapsto x * y$$

respectively. The graded additive group

$$A = \bigoplus_{r \geq 0} A(S_r)$$

becomes a graded ring with multiplication $\circ$. Moreover one has

$$(a+b) * c = a * c + b * c$$

$$(a \circ b) * c = (a * c) \circ (b * c)$$

$$(a * b) * c = a * (b * c)$$

$$b * 1 = b.$$

Here $1 \in B(S_o) = Z$.

Proof. The formal algebraic properties of these constructions follow by considering representatives once we have shown that there are well defined induced maps $\circ$ and $*$.

We factorise the required map $\circ$ as

$$A(S_r) \times A(S_t) \xrightarrow[p_1^* \times p_2^*]{} A(S_r \times S_t) \times A(S_r \times S_t)$$

$$\longrightarrow A(S_r \times S_t) \longrightarrow A(S_{r+t})$$

where p_1, p_2 are the projections, the second map is the multiplication in the ring $A(S_r \times S_t)$ and the third map is the extension homomorphism 5.12.1.

Similarly we factorise the map $*$ as

$$A(S_r) \times A(S_t) \xrightarrow[p_1^* \times w]{} A(S_r \int S_t) \times A(S_r \int S_t)$$

$$\longrightarrow A(S_r \int S_t) \longrightarrow A(S_{rt})$$

where p is the projection $S_r \int S_t \longrightarrow S_r$ and where w is induced by $Y \longmapsto Y^r$ (this well-defined!); the second map is again multiplication and the third extension.

We return to the map 5.13.5 which, obviously, is additive in the first variable, so that we obtain an action $A \times A(G) \longrightarrow A(G)$. Moreover the constructions of 5.13.8 have the following properties.

<u>Proposition 5.13.9.</u>

<u>For</u> $a_1, a_2 \in A$ <u>and</u> $b \in A(G)$

$$(a_1 \circ a_2) \cdot b = (a_1 \circ b)(a_2 \circ b)$$

$$(a_1 * a_2) \cdot b = a_1 \cdot (a_2 \cdot b) \quad .$$

The interpretation of these formulas is this: $a \in A$ induces an operation $b \longmapsto a \cdot b$ on $A(G)$. Addition and multiplication in A corresponds to pointwise addition and multiplication of operation. Finally $*$ is composition of operations. Hence A is a ring of operations. The operations have some obvious naturality properties which we do not write down. The proof of the identities is given by looking at representatives.

5.14. An example: The group SO(3).

Using Wolf [169], 2.6., one can see that SO(3) has the following conjugacy classes of subgroups:

$SO(3)$	
$S^1 \cong SO(1)$	maximal torus
$S \cong NS^1 \cong O(1)$	normalizer of S^1
$I = A_5$	icosahedral group
$O = S_4$	octahedral group
$T = A_4$	tetrahedral group
$D_n,\ n \geq 2$	dihedral group of order 2n
$Z/n,\ n \geq 1,$	cyclic group.

One has $ND_n = D_{2n}$, $n = 2$; $ND_2 = S_4$, $NA_4 = S_4$, $NS_4 = S_4$, $NA_5 = A_5$, $NO(1) = O(1)$. The cyclic groups do not have finite index in their normalizer.

The ring A(SO(3)) is the set of functions $z \in C(\phi, Z)$ such that

i) $z(H)$ arbitrary for $H = SO(3)$, A_5, S_4, NT.

ii) $z(D_n) \equiv z(D_{2n}) \bmod 2$, $n \neq 2$

iii) $z(A_4) \equiv z(S_4) \bmod 2$

iv) $z(S) \equiv z(S^1) \bmod 2$

v) $z(D_2) + 2z(A_4) + 3z(D_4) \equiv 0 \bmod 6$.

The continuity of z means $\lim_j z(D_{2^j n}) = z(S)$.

If H is a subgroup of SO(3) we denote for simplicity with the same symbol the element [G/H] of A(SO(3)). We give the multiplication table of the elements H. We put (k,n) for the greatest common divisor and let $d_{(k,n)} = 1$ if $(k,n) = k$ and $d_{(k,n)} = 0$ otherwise.

$(S)^2 = S + D_2$ $\qquad$ $S S^1 = S^1$

$S \cdot D_k = D_k + 2d_{(2,k)} D_2$ $\qquad$ $SI = D_5 + D_3 + D_2$

$S \cdot T = D_2$ $\qquad$ $SO = D_4 + D_3 + D_2$

$(S^1)^2 = 2S^1$ $\qquad$ $S^1 H = 0$ for $H \neq S, S^1$.

$$(D_k)^2 = 2D_k + 4d_{(2,k)} D_2$$

$$D_k D_n = 2D_{(k,n)} + 4d_{(2,(k,n))} D_2$$

$$D_k O = 2d_{(4,k)} D_4 + 2d_{(3,k)} D_3 + 2d_{(2,k)} (2-d_{(4,k)}) D_2$$

$$D_k T = 2d_{(2,k)} D_2$$

$$I^2 = I + T + D_5 + D_3$$

$$IT = 2T$$

$$IO = T + 2D_3 + D_2$$

$$O^2 = O + D_4 + D_3 + D_2$$

$$OT = T + D_2$$

$$T^2 = 2T$$

$$D_k I = 2d_{(5,k)} D_5 + 2d_{(3,k)} D_3 + 2d_{(2,k)} D_2$$

The ring A(SO(3)) contains the following idempotent elements

$$x = I - T - D_5 - D_3$$

$$y = S + O - D_4 - D_3$$

$x+y$, $1-x$, $1-y$, $1-x-y$.

5.15. Comments.

The general theory of the Burnside ring of a compact Lie group is based on the authors papers [64], [65], [66]. As far as the equivariant Euler characteristic is concerned there has been a parallel development in the cohomology of groups, see K. Brown [39], [40], [41]. We have been guided in 5.3 by Brown [39]. For 5.3.3 see Floyd ([83], III §3). For 5.3.4 see [39].

It would be desirable to give a unified treatment of the Burnside ring and results in Brown [39] . Also Bass [16] is relevant. The universal ring for Euler-Characteristic in 5.4 has been introduced by Oliver [118] and has also been used by Becker-Gottlieb 5.5.10 and 5.14 is due to Schwänzl [140] , 5.7 is an extension of work of Dress [79] . For general compact groups see Gordon [86] . It would be interesting to find a more general class of G-spaces which satisfy the relations between Euler-Characteristics 5.8.5; suitable finiteness conditions for cohomology should suffice. For 5.9.8 see Zassenhaus [171] and Raghunatan [130] . The results of 5.10 are based on the thesis of Gordon [86] ; see also Gordon [87] . The reader can see that a purely algebraic definition of the Burnside ring for finite torus extensions can be given. This algebraic definition is then also applicable to other arithmetic situations, e.g. representations over p-adic integers. If G acts on a disk D such that all D^H are either empty or contractible then D represents an idempotent in A(G). Oliver [118] has shown that essentially all idempotents of A(G) arise in this way. For 5.13 I could make us of an unpublished manuscript of Rymer [137] . For operations in the Burnside ring see also Siebeneicher [149] .

5.16. Exercises.

1. Compute the ring U(G) of 5.4 for G = SO(3).
2. Given a natural number $n \geq 2$. Can U(G) contain elements x such that $x^{n-1} \neq 0$, but $x^n = 0$?
3. Show that A(SO(3)) has three idecomposable idempotent elements.
4. Compute the units of A(SO(3)) and compare with the units obtainable from 1.5.3.
5. If G is cyclic then permutation representations given an isomorphism $A(G) \cong R(G;\mathbb{Q})$.

6. Use 5.13 to define a λ-ring structure on $A(G)$ by symmetric powers and show that the isomorphism of exercise 5 is compatible with λ-operations (but don't take exterior powers!).

7. Let S be any subring of the rationals. Determine the idempotent elements of $A(G) \otimes_Z S$, in particular for $S = Z_{(p)}$.

8. Let S_G be the homotopy category of pointed G-CW-complexes. Consider the Grothendieck group $K(S_G)$ of this category: The universal abelian group $S_G \longrightarrow K(S_G) : X \longmapsto [X]$, where each cofibration sequence $A \longrightarrow X \longrightarrow X/A$ gives rise to a relation $[X] = [A] + [X/A]$. Show that smashed-product $(X,Y) \mapsto X \wedge Y$ makes $K(S_G)$ into a commutative ring. Show that $K(S_G) \cong U(G)$.

6. Induction Theory.

In this section we present the formal theory of induced representations, restriction homomorphisms, transfer maps. This axiomatic theory was developed mainly by Green [88] and Dress [80] , [81] . The basic axioms are abstract forms of the Frobenius reciprocity law and the Mackey double coset formula of ordinary representation theory. Later we shall apply the formalism to equivariant homology, cohomology, and topological transfer maps.

6.1. Mackey functors.

Let G be a finite group and let $G^{\wedge}$ or G-Set be category of finite G-sets and G-maps. Let Ab be the category abelian groups.

A bi-functor

$$M = (M^*, M_*) : G\text{-Set} \longrightarrow Ab$$

consists of a contravariant functor $M^* : G\text{-Set} \longrightarrow Ab$ and a covariant functor $M_* : G\text{-Set} \longrightarrow Ab$; the functors are assumed to coincide on objects. We write

$$M(S) = M_*(S) = M^*(S)$$

for a finite G-set S. If $f : S \longrightarrow T$ is a morphism we often use the notation

$$M_*(f) = f_*, \quad M^*(f) = f^*.$$

We use the topological notation: a lower star for covariant functors ("homology"). Dress unfortunately uses a different notation.

A bi-functor $M = (M^*, M_*)$ is called a <u>Mackey functor</u> if it has the following properties:

(6.1.1) For any pullback diagram in G-Set

$$\begin{array}{ccc} U & \xrightarrow{\quad F \quad} & S \\ {\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ T & \xrightarrow{\quad f \quad} & V \end{array}$$

the diagram

$$\begin{array}{ccc} M(U) & \xrightarrow{\quad F_* \quad} & M(S) \\ {\scriptstyle H^*}\big\uparrow & & \big\uparrow{\scriptstyle h^*} \\ M(T) & \xrightarrow{\quad f_* \quad} & M(V) \end{array}$$

is commutative.

(6.1.2) The two embeddings $S \longrightarrow S + T \longleftarrow T$ into the disjoint union define an isomorphism

$$M^*(S+T) \longrightarrow M^*(S) \oplus M^*(T).$$

Let M and N be bi-functors. A <u>natural transformation of bi-functors</u> $X : M \longrightarrow N$ consists of a family of maps $X(S) : M(S) \longrightarrow N(S)$, indexed by the objects of G-Set, such that this family is a natural transformation $M_* \longrightarrow N_*$ and $M^* \longrightarrow N^*$.

Let M be a Mackey functor and S a G-set. Then

$$M_S : T \longrightarrow M(S \times T)$$

$$M_S^*(f) = M^*(id_S \times f),\ M_{S*}(f) = M_*(id_S \times f)$$

defines a Mackey functor M_S, as one easily checks. The projection maps $pr : S \times T \longrightarrow T$ define natural transformation of bi-functors

$$\Theta^S : M \longrightarrow M_S \quad , \quad \Theta^S(T) = pr^*$$

$$\Theta_S : M_S \longrightarrow M \quad , \quad \Theta_S(T) = pr_* .$$

The relevant commutative diagrams follow from the functor properties of M and from 6.1.1.

The functor M is called S-<u>injective</u> (S-<u>projective</u>) if Θ^S (Θ_S) is split-injective (split-surjective) as a natural transformation of bi-functors.

<u>Proposition 6.1.3.</u>

<u>Let</u> M <u>be a Mackey functor</u>. <u>Then the following assertions are equivalent:</u>

i) M <u>is</u> S-<u>injective</u>.

ii) M <u>is</u> S-<u>projective</u>.

iii) M <u>is a direct summand of</u> M_S <u>as bi-functor</u>.

<u>Proof.</u>

i) $\Rightarrow$ iii) By definition of S-injectivity.

iii) $\Rightarrow$ i) The assumption of iii) is that we have natural transformation $\Theta : M \longrightarrow M_S$, $\Psi : M_S \longrightarrow M$ such that $\Psi\Theta = id$. We have to find a natural transformation $\psi^S : M_S \longrightarrow M$ such that $\psi^S\Theta^S = id$. For a G-set T we define $\psi^S(T)$ by the following diagram

$$
\begin{array}{ccccc}
M(T) & \xrightarrow[\Theta^S(T)]{pr^*} & M(S \times T) & \overset{S}{\dashrightarrow}{}_{(T)} & M(T) \\
\Big\downarrow \Theta(T) & & \Big\downarrow \Theta(S\times T) & & \Big\uparrow (T) \\
M(S \times T) & \xrightarrow[pr^*]{} & M(S \times S \times T) & \xrightarrow[(d \times id)^*]{} & M(S \times T)
\end{array}
$$

where $d : S \longrightarrow S \times S$ is the diagonal. The left square is commutative by naturality. Using $(d \times id)^* pr^* = id$ and $\Psi(T)\Theta(T) = id$ one proves $\Psi^S(T)\Theta^S(T) = id$. Moreover Ψ^S is defined as a composition of three natural transformation of bi-functors hence itself such a natural transformation.

ii) <=> iii) is proved analogously.

Let S be a G-set. We let S^o be a point and $S^k = \prod_{i=o}^{k-1} S$. We denote $pr_i : S^{k+1} \longrightarrow S^k$ the projection which omits the i-th factor, $0 \leq i \leq k$. If M is a Mackey functor we have two chain complexes

(6.1.4) $$0 \longrightarrow M(S^o) \xrightarrow{d^o} M(S^1) \xrightarrow{d^1} M(S^2) \xrightarrow{d^2} \cdots$$

(6.1.5) $$0 \longleftarrow M(S^o) \xleftarrow[d_o]{} M(S^1) \xleftarrow[d_1]{} M(S^2) \xleftarrow[d_2]{} \cdots$$

defined by $d^k = \sum_{i=o}^{k} (-1)^i \, pr_i^*$, $d_k = \sum_{i=o}^{k} (-1)^i \, pr_{i*}$.

<u>Proposition 6.1.6.</u>

<u>Let</u> M <u>be</u> <u>a</u> <u>Mackey</u> <u>functor</u>. <u>Then</u>

i) M_S <u>is</u> <u>always</u> S-<u>injective</u> <u>and</u> S-<u>projective</u>.

ii) <u>If</u> M <u>is</u> S-<u>injective</u> <u>then</u> <u>the</u> <u>complexes</u> 6.1.4 <u>and</u> 6.1.5 <u>are</u> <u>exact</u>.

Proof.

The splitting of $M_S \longrightarrow (M_S)_S$ appears in the proof of 6.1.3. Let Ψ be a splitting of Θ^S. Then a contracting homotopy of 6.1.4 is given by the maps

$$s^{k+1} := \Psi(S^k) : M(S \times S^k) \longrightarrow M(S^k)$$

A splitting of Θ_S gives a contracting homotopy for 6.1.5.

Remark.

Instead of using functors into Ab one can consider functors into the category of modules over a ring or into an abelian category. This remark also applies to subsequent developments.

It is often convenient to denote $M(G/H)$ by $M(H)$. If $H < K < G$ and $f : G/H \longrightarrow G/K$ the canonical map then

$$f \ : M(K) = M(G/K) \longrightarrow M(G/H) = M(H)$$

is called restriction from K to H

$$res_H^K$$

and

$$f_* : M(H) = M(G/H) \longrightarrow M(G/K) = M(K)$$

is called induction from H to K

$$ind_H^K$$

The axioms for a Mackey functor essentially tell how res and ind behave under composition. This is the so called double coset formula which one can never remember and which is avoided by this axiomatic treatment. Let

$$\begin{array}{ccc} G/H \times G/K & \xrightarrow{P} & G/K \\ \Big\downarrow Q & & \Big\downarrow q \\ G/H & \xrightarrow{p} & G/G \end{array}$$

be the canonical pullback. The orbits $A_1, \ldots, A_r$ of $G/H \times G/K$ correspond to the double cosets $H \backslash G/K$. Let $P(i)$, $Q(i)$ be the restriction of P, Q to A_i. Then 6.1.1 and 6.1.2 say

$$(6.1.7) \qquad \mathrm{res}_H^G \, \mathrm{ind}_K^G = \sum_{i=1}^{r} P(i)_* \, Q(i)^*.$$

If A_i is the orbit through $(1,x)$ then via $A_i = G/G_{(1,x)}$

$$(6.1.8) \qquad Q(i)^* = \mathrm{res}^H_{H \cap x K x^{-1}}$$

and

$$(6.1.9) \qquad P(i)_* = \mathrm{ind}^K_{K \cap x^{-1} H x} \circ c(x)_*$$

where $c(x)$ is conjugation $g \longmapsto x^{-1} g x$. The double coset formulas 6.1.7 - 6.1.9 are sufficient to reconstruct the whole Mackey functor.

Similar remarks apply to the exact sequences 6.1.4 and 6.1.5. We spell out what the exactness of 6.1.4 at $M(S^1)$ means in terms of double cosets. Let $S = \coprod_{H \in F} G/H$, where F is a family of subgroups of G. Then $M(S) = \bigoplus_{H \in F} M(G/H)$. The image of $M(S^0)$ in $M(S)$ is equal to the difference kernel of the two projection maps $p_i^* : M(S) \longrightarrow M(S \times S)$ which are maps

$$\bigoplus_{H \in F} M(G/H) \longrightarrow \bigoplus_{(H,K) \in F \times F} M(G/H \times G/K)$$

Then $(x_H) \in \bigoplus_{H \in F} M(G/H)$ is in the kernel if and only if for each $x \in K$ and $(H,K) \in F \times F$ $\mathrm{res}(x_H) \in M(H \cap x Kx^{-1})$ is equal to $\mathrm{res} \circ c(x)x_K$, where again $c(x)$ is the map induced by the conjugation $x^{-1}Hx \cap K \longrightarrow xKx^{-1} \cap H$. It is seen that this difference kernel is actually an inverse limit.

6.2. Frobenius functors and Green functor.

Let M,N, and L be Mackey functors G-Set $\longrightarrow$ Ab. A pairing

$$M \times N \longrightarrow L$$

is a family of bilinear maps

$$M(S) \times N(S) \longrightarrow L(S) : (x,y) \longmapsto x \cdot y$$

indexed by the objects of G-Set, such that for any morphism $f : S \longrightarrow T$ the following holds

$$\begin{aligned} L^* f(x\cdot y) &= (M^* fx)\cdot(N^* fy) , && x \in M(T) , && y \in N(T) \\ x\cdot(N_* fy) &= L_* f((M^* fx)\cdot y) , && x \in M(T) , && y \in N(S) \qquad (6.2.1) \\ (M_* fx)\cdot y &= L_* f(x\cdot(N^* fy)) , && x \in M(S) , && y \in N(T). \end{aligned}$$

These formulas make sense if M,N, and L are just bi-functors.A bi-functor F together with a pairing $F \times F \longrightarrow F$ is called a Frobenius functor if $F(S) \times F(S) \longrightarrow F(S)$ makes $F(S)$ into an associative ring with unit and morphisms f^* preserve units.

A Green functor U : G-Set $\longrightarrow$ Ab is a Mackey functor U together with a pairing $U \times U \longrightarrow U$ making it into a Frobenius functor.

If U is a Green functor then a <u>left</u> U-<u>module</u> is a Mackey functor M together with a pairing $U \times M \longrightarrow M$ such that via this pairing M(S) becomes a left U(S)-module (the unit $1_{U(S)} \in U(S)$ acting as identity).

<u>Theorem 6.2.2.</u>

<u>Let</u> U : G-Set $\longrightarrow$ Ab <u>be a Green functor</u>. <u>Let</u> S <u>be a</u> G-<u>set</u>. <u>Then the following assertions are equivalent</u>:

i) <u>The map</u> $f_* : U(S) \longrightarrow U(P)$ <u>is surjective</u> (P = Point).

ii) U <u>is</u> S-<u>injective</u>.

iii) <u>All</u> U-<u>modules are</u> S-<u>injective</u>.

<u>Proof.</u>

iii) $\Rightarrow$ ii), because U is a U-module.

ii) $\Rightarrow$ i), because by 6.1.3. U is S-projective; in particular $U_S(P) \longrightarrow U(P)$ is split surjective.

i) $\Rightarrow$ iii): Choose $x \in U(S)$ with $f_*(x) = 1$. Let M be a U-module. Define a natural transformation $\psi : M_S \longrightarrow M$ by

$$\psi(T) : M(S \times T) \longrightarrow M(T) : y \longmapsto q_*(p^*x \cdot y)$$

where $p : S \times T \longrightarrow S$ and $q : S \times T \longrightarrow T$ are the two projections. One checks that ψ is a natural transformation of Mackey functors. Moreover ψ is left inverse to $\theta^S : M \longrightarrow M_S$ because for $z \in M(T)$ one has by 6.2.1

$$\psi\theta^S(T)(z) = q_*(p^*x \cdot q^*y) = (q_*p^*x) \cdot y$$

and by 6.1.1. $q_*p^*x = g^*f_*x = g^*1 = 1$, where we have used the pullback diagram

$$\begin{array}{ccc} S \times T & \xrightarrow{\;q\;} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow[\;f\;]{} & P \end{array}$$

The universal example of a Green functor is the <u>Brunside ring functor</u>. We describe this aspect of the Burnside ring now. Let $A[S]$ be the Burnside ring of finite G-sets over S. If $f : S \longrightarrow T$ is a morphism then pullback along f defines a ring homomorphism $f^*: A[T] \longrightarrow A[S]$ and composition with f defines an additive map $f_* : A[S] \longrightarrow A[T]$. The ring structure on $A[S]$ defines the pairing $A \times A \longrightarrow A$. It is easily checked that these data make A into a Green functor. (Compare 5. where we have studied a slightly more general situation.)

<u>Proposition 6.2.3.</u>

<u>Let</u> M <u>be a Mackey functor</u>. <u>Then</u> M <u>is in a canonical way a module over the Burnside ring functor</u>.

<u>Proof.</u>

Given $f : T \longrightarrow S$ we consider the homomorphism $f_* f^* : M(S) \longrightarrow M(S)$. The assignment $(f,x) \longmapsto f_* f^* x$ is additive in f and induces therefore a bilinear map $A[S] \times M(S) \longrightarrow M(S)$. We leave it as an exercise to verify that this defines a pairing and makes M into an A-module.

Let U be a Green functor. The assignment $f : T \longrightarrow S \longmapsto f_* f^* 1_S$ induces a ring homomorphism

(6.2.4) $$h(S) = h : A[S] \longrightarrow U(S)$$

and the h(S) from a natural transformation of Green functors. This generalizes permutation representations.

We now discuss defect sets.

Proposition 6.2.4.

Let X *and* Y *be finite* G-*sets and let* U *be a Green functor. Then* $U(X) \longrightarrow U(P)$ *and* $U(Y) \longrightarrow U(P)$ *are surjective if and only if* $U(X \times Y) \longrightarrow U(P)$ *is surjective.* (P = Point.)

Proof.

If $U(X \times Y) \longrightarrow U(P)$ is surjective we see from the factorization $U(X \times Y) \longrightarrow U(X) \longrightarrow U(P)$ that $U(X) \longrightarrow U(P)$ is surjective. If $U(Y) \longrightarrow U(P)$ is surjective then U is Y-projective so that $U(Y \times X) \longrightarrow U(X)$ is surjective for any X.

Corollary 6.2.5.

There exists a unique minimal set D(U) *of conjugacy classes of subgroups of* G *such that the sum of the induction maps* $U(H) \longrightarrow U(G)$, $(H) \in D(U)$ *is surjective.*

D(U) is called the *defect set* of the Green functor U. The famous induction theorem of Brauer is in this terminilogy the statement that the defect set of the complex representation ring are the groups S x P, P a p-group, S cyclic.

6.3. Hyperelementary induction.

An induction theorem for a given Mackey functor is a theorem which computes its defect base or gives at least some restrictions on the defect base. We shall present one general result of this nature.

We begin with a result about restriction and induction for the

Burnside ring. Let N be a family of subgroups of G which is closed with respect to subgroups and conjugation. Let p a prime and define

$$(6.3.1) \qquad N^p = \{ H < G \mid \exists K \triangleleft H \text{ with } K \in N \text{ and } |H/K| \text{ a power of } p \} .$$

Let an index (p) denote localization at the prime ideal (p). Let Ke(N) denote the kernel of the restriction maps $A(G)_{(p)} \longrightarrow \prod_{H \in N} A(H)_{(p)}$ and let $Im(N^p)$ denote the image of the sum of the induction maps $\bigoplus_{H \in N^p} A(H)_{(p)} \longrightarrow A(G)_{(p)}$. Then we have

<u>Proposition 6.3.2.</u>

$Ke(N) + Im(N^p) = A(G)_{(p)}$.

<u>Proof.</u>

$Ke(N) + Im(N^p)$ is an ideal of $A(G)_{(p)}$ because Ke(N) certainly is an ideal as a kernel of a ring homomorphism and for any Frobenius functor the image of an induction map is an ideal (use 6.2.1). If this ideal is different from $A(G)_{(p)}$ then there exists a maximal ideal q of $A(G)_{(p)}$ with $Ke(N) + Im(N^p) \subset q$. This ideal q has the form q = q(L,p), see 5. 7. 2 . Since $Ke(N) \subset q$ this ideal extends to $\prod_{H \in N} A(H)$ (use e. g. Atiyah - Mac Donald [11] , 5.10), i. e. we may assume q = q(L,p) with $L \in N$. By 5. 7. 1 q(L,p) = q(K,p) where $G/K \notin q$ and by 5. 7. 3 $K \in N^p$. Hence G/K is the image of 1 under the induction map $A(K) \longrightarrow A(G)$. But $G/K \notin q$ contradicts $G/K \in Im(N^p) \subset q$. Hence a q with $Ke(N) + Im(N^p) \subset q$ cannot exist.

Let now U be a Green functor G-Set $\longrightarrow$ Ab. As usual we denote U(G/H) for the G-set G/H by U(H). Let N and N^p be as above.

<u>Theorem 6.3.3.</u>

<u>Assume that any torsion element in</u> U(G) <u>is nilpotent</u>. <u>Assume that the</u>

restriction map

$$U(G) \otimes \mathbb{Q} \longrightarrow \prod_{H \in N} U(H) \otimes \mathbb{Q}$$

is injective. Then the induction map

$$\bigoplus_{H \in N^p} U(H)_{(p)} \longrightarrow U(G)_{(p)}$$

is surjective.

Proof.

The injectivity and nilpotency hypothesis of the theorem imply that any element in the kernel of $U(G)_{(p)} \longrightarrow \prod_{H \in N} U(H)_{(p)}$ is nilpotent. By 6.3.2 we find $x \in Ke(N)$, $y \in Im(N^p)$ with $x + y = 1 \quad A(G)_{(p)}$. Now apply the natural transformation $h : A(G)_{(p)} \longrightarrow U(G)_{(p)}$ of 6.2.4. Then $h(x) + h(y) = 1 \in U(G)_{(p)}$ and $h(x)$, contained in the kernel of $U(G)_{(p)} \longrightarrow \prod_{H \in N} U(H)_{(p)}$, is nilpotent. Therefore $h(y) = 1 - h(x)$ is a unit. But $h(y)$ is in the image of $\bigoplus_{H \in N^p} U(H)_{(p)} \longrightarrow U(G)_{(p)}$, so that this image being an ideal must be all of $U(G)_{(p)}$.

If $N = C$ is family of cyclic subgroups of G, then N^p is the family of p-hyperelementary subgroups of G. A subgroup is called hyperelementary if it is p-hyperelementary for some prime p. Let Hy be the class of hyperelementary subgroups of G.

Corollary 6.3.4.

If U(G) is torsion free and $U(G) \longrightarrow \prod_{H \in C} U(H)$ is injective then U satisfies hyperelementary induction, i. e. the induction map

$$\bigoplus_{H \in Hy} U(H) \longrightarrow U(G)$$

is surjective.

A particular example where the hypothesis of 6.3.4 is fulfilled is the Green functor "rational representation ring". By 6.2.2. any module over this Green functor also satisfies hyperelementary induction.

6.4. Comments.

This section is based on Dress [80] , [81] . We refer to these papers for further details, in particular for the connection with classical induction theorems. The reader should also study Dress [80] , § 7 in order to see a general construction of Mackey functors which works in most of the algebraic applications. As a research problem I suggest that the reader takes the double coset formula of 5.12 and develops induction theory for compact Lie groups in analogy to the theory in this section. For applications of induction theory in topology see the next section (also for compact Lie groups).

6.5. Exercises.

1. Make multiplicative induction (5.12) as part of a Mackey functor.
2. Let $(p) \subset Z$ be a prime ideal. What is the defect set of the localized Burnside ring functor $A(G)_{q(H,p)}$?
3. Provide the details in the proof of 6.2.3.

7. Equivariant homology and cohomology.

We describe localization and splitting theorems for equivariant homology and cohomology theories. In particular we use the fact that such theories are modules over the Burnside ring. We compute localizations at prime ideals of the Burnside ring. Our treatment in this chapter is axiomatic.

7.1. A general localization theorem.

Let G be a compact Lie group. A G-equivariant cohomology theory consists of a contravariant, G-homotopy invariant functor $h_G^*(?,?)$ from a suitable category of pairs of G-spaces (e. g. compact spaces, or G-CW-complexes) into graded abelian groups. The grading is by an abelian group A which may be the integers, the real representation ring or some subquotient of it, etc. It assumed that A is equipped with a homomorphism $i : Z \longrightarrow A$ so that expressions like $a+i(n) = a+n$, $a \in A$, $n \in Z$, make sense. We require the long exact cohomology sequence to hold (at least for closed G-cofibrations $A \subset X$) and the suspension isomorphism $\tilde{h}_G^*(X) \cong \tilde{h}_G^{*+1}(SX)$. In the following we gradually add more and more axioms, like suspension isomorphisms for representations, product structure, continuity etc.

If H is a subgroup of G we write

$$(7.1.1) \qquad h_H^*(X,Y) = h_G^*(G\times_H X,\ G\times_H Y)$$

for a pair (X,Y) of H-spaces and consider $h_H(?,?)$ as H-equivariant cohomology theory.

Let now $k_G^*(?,?)$ be another equivariant cohomology theory with the same grading as h_H^* and which is multiplicative. In particular $k_G^*(X)$ is a graded-commutative ring with unit. We assume given a pairing

$$k_G^*(X,Y) \times h_G^*(X,Y) \longrightarrow h_G^*(X,Y)$$

of cohomology theories which makes $h_G^*(X,Y)$ a $k_G^*(X,Y)$-module. In particular $h_G^*(G\times_H X)$ is via the projection $p : G\times_H X \longrightarrow G/H$ and $k_G^*(G/H) \longrightarrow h_G^*(G/H)$ an $k_G^*(G/H)$-module. Moreover it is also via $k_G^* \longrightarrow k_G^*(G\times_H X) \longrightarrow h_G^*(G\times_H X)$ an $k_G^* = k_G^*(\text{Point})$ module and this module structure "factors" over the ring homomorphism $k_G^* \longrightarrow k_G^*(G/H) = k_H^*$, called restriction homomorphism.

Let $S \subset k_G^*$ be a multiplicatively closed subset which (for simplicity) lies in the center of k_G^*, and is in particular commutative in the ungraded sense (center also in the ungraded sense). Let X be a G-space and put

(7.1.2) $$X^S = \{ x \in X \mid S \cap \text{Kernel}(k_G^* \longrightarrow k_G^*(G/G_x)) = \emptyset$$.

<u>Proposition 7.1.3. Let</u> X <u>be a compact</u> G-<u>space with</u> $X^S = \emptyset$. <u>Then the localization</u>

$$S^{-1} h_G^*(X) = 0.$$

(Graded localization. Elements of S are made invertible.)

<u>Proof.</u> Given $x \in X$ we can find by the slice theorem (Bredon [37], II 5.4) a G-neighbourhood U of the orbit Gx and a G-map $r : U \longrightarrow G/G_x$. If $U_o = r^{-1}(G_x)$ then canonically $U = G \times_{G_x} U_o$ and r is the G-extension of $U_o \longrightarrow$ Point. Since x does not lie in X^S we can find $s \in S$ which is contained in the kernel of $k_G^* \longrightarrow k_G^*(G/G_x)$. Since the k_G^*-module structure of $h_G^*(U)$ factors over $k_G^* \longrightarrow k_G^*(G/G_x)$ we see that $s h_G^*(U) = 0$,

hence $S^{-1}h_G^*(U) = 0$. Covering X by a finite number of such U, using the Mayer - Vietoris sequence and the exactness of the localization functor we conclude that $S^{-1}h_G^*(X) = 0$.

We now consider compact G-spaces X in general. If V is a compact G-neighbourhood of X^S in X then by excision and 7.1.3 we have $S^{-1}h_G^*(X,V)=0$. Now assuming either the continuity

$$\text{(7.1.4)} \qquad \operatorname*{colim}_V h_G^*(X,V) = h_G^*(X,X^S)$$

of the cohomology theory or local properties of the pair X, X^S which imply 7.1.4 (e. g. neighbourhood retract). We obtain

<u>Proposition 7.1.5.</u> <u>Let</u> X <u>be</u> <u>a</u> <u>compact</u> G-<u>space</u> <u>such</u> <u>that</u> 7.1.4 <u>holds</u>. <u>Then</u> <u>the</u> <u>inclusion</u> $X^S \longrightarrow X$ <u>induces</u> <u>an</u> <u>isomorphism</u>

$$S^{-1}h_G^*(X) \cong S^{-1}h_G^*(X^S) \ .$$

There are many variants of 7.1.3 and 7.1.5 according to the different technical (axiomatic) assumptions about theories and spaces involved. We mention some of them. First of all the treatment of homology is quite analogous. Compactness of the space in 7.1.3 may be replaced by finite dimensionality, working with the spectral sequence of a covering and an additive theory.

We now describe a particular of the localization process. We assume that our cohomology theory h_G has suspension isomorphisms for a suitable set of representations, i. e.: Given a family $(V_j \mid j \in J)$ of complex representations and to each j a natural isomorphism $s_j : \tilde{h}_G^*(X) \cong \tilde{h}_G^{*+|j|}(V_j^c \wedge X)$ where $V_j^c = V_j \cup \infty$ is the one-point-compactification and $|j|$ is a suitable index depending additively on V_j (e. g.

the dimension or V_j itself). We assume for simplicity that the representations are complex in order to avoid sign problems. The s_j are assumed to commute. We define the <u>multiplication with the Euler class</u> of V_j to be the composite map

$$\tilde{h}_G^*(X) \xrightarrow[s_j]{} \tilde{h}_G^{*+|j|}(V_j^c \wedge X) \longrightarrow \tilde{h}_G^{*+|j|}(X)$$

where the second map is induced by inclusion of the zero in V_j. Actually this is a special case of the previously discussed module structure, coming from a natural transformation of stable equivariant cohomotopy into h_G. Let S be the multiplicatively closed subset generated by all such Euler-classes. Then $X \setminus X^S$ is the set of all orbits which can be mapped into $V \setminus \{o\}$, where V is any finite direct sum of V_j's. If in particular V_j consists of all non-trivial irreducible representations then X^S is the fixed point set of X. (See tom Dieck [56] for further information on this construction.)

<u>7.2. Classifying spaces for families of isotropy groups.</u>

Let G be a compact Lie group. A set F of closed subgroups is called a <u>familiy</u> if it is closed under conjugation and taking subgroups. (For some of the following investigations it suffices: closed under conjugation and intersection).

Let F be a family. A G-space X is called F-<u>trivial</u> if there exists a G-map $X \longrightarrow G/H$ for some $H \in F$. The G-space X is called F-<u>numerable</u>, if there exists a numerable covering $(U_j \mid j \in J)$ of X by F-trivial G-subsets. See Dold [71] for the notion of numerable covering. Partitions of unity in our context should consist of G-invariant functions.

Let F be a family. We denote by T(G,F) the category of F-numerable G-spaces. The isotropy groups of such spaces lie in F. Let T(G,F)h be the corresponding homotopy category.

<u>Proposition 7.2.1.</u> <u>The</u> <u>category</u> T(G,F)h <u>contains</u> <u>a</u> <u>terminal</u> <u>object</u> E(F), <u>i. e.</u> <u>an</u> <u>object</u> E(F) <u>such</u> <u>that</u> <u>each</u> F-<u>numerable</u> G-<u>space</u> X <u>admits</u> <u>a</u> G-<u>map</u> $X \longrightarrow E(F)$ <u>unique</u> <u>up</u> <u>to</u> G-<u>homotopy</u>.

<u>Proof.</u> We immitate the Milnor construction [115] of a universal bundle. There exists a countable system $(H_j \mid j \in J)$ of groups $H_j \in F$ such that every group in F is conjugate to an H_j (Palais [124] , 1.7.27) Let $E_j = G/H_j * G/H_j * \ldots$ be the join of a countibly infinite number of copies G/H_j. Let $E(F) = *_{j \in J} E_j$ be the join of the E_j (always carrying the Milnor topology).

Let X be an object of T(G,F). We choose a numerable covering $(U_a \mid a \in A)$ by G-sets $U_a \subset X$ and G-maps $f_a : U_a \longrightarrow G/H_a$ with $H_a \in \{H_j \mid j \in J\}$. One can assume that A is countable (compare tom Dieck [49] , Hilfssatz 2). From $(U_a \mid a \in A)$ and a subordinate G-invariant partition of unity one constructs a G-map $X \longrightarrow E(F)$ and shows as in tom Dieck [49] that any two G-maps are G-homotopic. The space E(F) is contained in T(G,F) (see Dold [71] , 8. for numerability). Hence E(F) is the desired terminal object.

<u>Remark 7.2.2.</u> A terminal object of T(G,F) is uniquely determined up to G-homotopy equivalence. If F_∞ is the family of all subgroups of G then $E(F_\infty)$ is G-contractible because a point is a terminal object in $T(G,F_\infty)h$.

<u>Proposition 7.2.3.</u> <u>Let</u> X <u>be</u> <u>an</u> <u>object</u> <u>in</u> T(G,F). <u>Then</u> $X * E(F)$ <u>is</u> G-<u>homotopy-equivalent</u> <u>to</u> E(F).

<u>Proof.</u> By the methods of tom Dieck [49] one proves that any two G-

maps $Z \longrightarrow Y$ for

$$Y = E(F) * X * X * \ldots .$$

are G-homotopic. If $X \in T(G,F)$ then $Y \in T(G,F)$, so that Y is a terminal object in $T(G,F)h$. This yields the G-homotopy equivalences

$$E(F) * X \simeq Y * X \simeq Y \simeq E(F) \ .$$

Let H be a subgroup of G. For a G-space X let $\mathrm{res}_H X$ be the H-space obtained by restricting the group action. If F is a family of subgroups of G let $F/H = \{L \cap H \mid L \in F\}$ be the induced family of subgroups of H. With these notations we have

Proposition 7.2.4. $\mathrm{res}_H\, E(F) = E(F/H)$.

Proof. By adjointness

$$[Y,\ \mathrm{res}_H\, E(F)]_H = [G \times_H Y,\ E(F)]_G .$$

If $Y \in T(H,F/H)$ then $G \times_H Y \in T/G,F)$. Hence the H-equivariant homotopy set $[Y,\ \mathrm{res}_H\, E(F)]_H$ contains a single element which means that $\mathrm{res}_H E(F)$ is a terminal object. Note that $\mathrm{res}_H E(F) \in T(H,F/H)$.

7.3. Adjacent families.

Families of isotropy groups have been used successfully in bordism theory by Conner and Floyd [47] and later by Stong [155] , Kosniowski [106] and others. The classifying spaces E(F) of 7.2 allow to extend some of these methods to arbitrary equivariant homology and cohomology theories. We give some indications of how this can be done.

Let $F_1 \subset F_2$ be two families of subgroups of G and let E_i by a terminal object of $T(G,F_i)h$. Then we have a G-map $f : E_1 \longrightarrow E_2$ unique up to G-homotopy. In the following we assume f to be a closed G-cofibration (replace, if necessary, E_2 with the mapping cylinder of f). If $f' : E_1' \longrightarrow E_2'$ is another such G-cofibration then the pair (E_2,E_1) is G-homotopy-equivalent to the pair (E_2',E_1'); compare tom Dieck-Kamps-Puppe [70] , Satz 2.32. The G-homotopy equivalence moreover is unique in the category of pairs (use terminality).

Suppose an equivariant homology theory h_* is given. We define a new homology theory by

$$(7.3.1) \qquad h_*[F_2,F_1](X,Y) := h_*(E_2 \times X,\ E_1 \times X \cup E_2 \times Y).$$

The exact homology sequence of a pair follows without trouble if Y is closed in X (or use mapping cylinders). Another choice of (E_2,E_1) yields, by the remarks above, a functor which is canonically isomorphic to $h_*[F_2,F_1]$. We put $h_*[F_2,\emptyset] = h_*[F_2]$ if F_1 is empty, i. e.

$$h_*[F_2](X,Y) := h_*(E_2 \times X,\ E_1 \times X).$$

The exact homology sequence of the triple

$$(E_2 \times X,\ E_1 \times X \cup E_2 \times Y,\ E_2 \times Y)$$

gives via the excision isomorphism

$$h_*(E_1 \times X,\ E_1 \times Y) \cong h_*(E_1 \times X \cup E_2 \times Y,\ E_2 \times Y)$$

the long exact sequence of homology theories

$$(7.3.2) \quad \dots \longrightarrow h_{n+1}[F_2,F_1](X,Y) \longrightarrow h_n[F_1](X,Y) \longrightarrow$$
$$\longrightarrow h_n[F_2](X,Y) \longrightarrow \dots$$

where n again is taken from a suitable index set.

The relation of the homology theories to the exact sequences of Conner and Floyd is as follows. (We use the notations of Stong [155] .) Let $\mathfrak{N}_*^G(F_2,F_1)$ be the unoriented G-bordism theory of manifolds in $T(G,F_2)$ with boundary in $T(G,F_1)$. Then

Proposition 7.3.3. There exists a natural isomorphism

$$\mathfrak{N}_*^G(F_2,F_1) \cong \mathfrak{N}_*^G[F_2,F_1].$$

Proof. Exercise. (See tom Dieck [57] .)

Proposition 7.3.3 tells us that bordism with families is unrestricted bordism of suitable spaces.

One of the main uses of families is the induction over orbit types using adjacent families. Two families $F_2 \supset F_1$ are called adjacent if their difference $F_2 \setminus F_1$ is just a single conjugacy class. We are going to analyze this situation.

Let $F_2 \supset F_1$ be adjacent, differing by the conjugacy class of H. Let CZ denote the cone over the space Z. Then we have

Proposition 7.3.4. There exists a canonical natural isomorphism

$$h_*[F_2,F_1](X,A) \cong h_*(G\times_{NH}E(NH/H) \times (CEF_2,EF_2) \times (X,A)).$$

Proof. In the statement of the proposition E(NH/H) is of course the free numerable NH/H-space. One shows that

$$(G\times_{NH} E(NH/H)) * EF_1$$

is a terminal object of $T(G,F_2)h$, hence can be taken as space EF_2. To prove this one recondires the proof of 7.2.1. The above claim then follows from the following considerations: If A and B are G-spaces and P is a point, then we have a G-homeomorphism

$$A * B \cong (A * P) \times B \cup A \times (B * P).$$

Using excision this yields

$$\begin{aligned} &h_*(A * B, B) \cong \\ &h_*((A * P) \times B \cup A \times (B * P), B) \cong \\ &h_*((A * P) \times B \cup A \times (B * P),\ (A * P) \times B) \cong \\ &h\ (A \times (B * P),\ A \times B). \end{aligned}$$

Moreover the pair $(B * P,\ B)$ is G-homotopy-equivalent to the pair (CB,B).

7.4. Localization and orbit families.

We assume given an additive G-homology theory h_* which is stable in the following sense: Let V be a complex G-module. Then we are given suspension isomorphism as in 7.1

$$s_V : \tilde{h}_*(X) \cong \tilde{h}_{*+|V|}(V^c \wedge X)$$

which are compatible $s_W s_V = s_{W \oplus V}$. We assume that the theory is multiplicative with unit $1 \in \tilde{h}_o(S^o)$. The image of 1 under

$$\tilde{h}_o(S^o) \xrightarrow[n_*]{} \tilde{h}_o(V^c) \xleftarrow[\cong]{} \tilde{h}_{-|V|}(S^o)$$

is called Euler class $e(V)$ of V (n is the zero section $S^o = \{o, \infty\} \to V^c$).
Let M be a set of G-modules which is closed under direct sums. Let

$$S = S(M) = \{e(V) \mid V \in M\}.$$

We formally invert the elements of S and obtain a new homology theory

$$S^{-1}\, h_*(X,A).$$

Theories of this type where investigated e. g. in tom Dieck [56], [53], [58], [59].

Let F_∞ be the family of all subgroups of G. Let F_S be the family of isotropy groups appearing on unit spheres $S(V)$, $V \in M$. Then we have

<u>Proposition 7.4.1. There exists a natural isomorphism of homology theories</u>

$$S^{-1}\, h_*(X,A) \cong h_*[F_\infty, F_S](X,A) .$$

<u>Proof.</u> As in tom Dieck [56] one sees that $S^{-1}\, h_*(X,A)$ is a direct limit over groups $h_*((DV,SV) \times (X,A))$ where V runs through the G-modules in M. Since an additive homology theory is compatible with direct limits we have to show essentially the following: Let V_∞ be the direct sum of a countable number of all irreducible representations which appear as direct summands in modules of M. Then the unit sphere $S(V_\infty)$ is a terminal object in $T(G,F_S)h$. Obviously $S(V_\infty) \in F(G,F_S)$. Any two G-maps $S(V_\infty) \longrightarrow S(V_\infty)$ are G-homotopic (Husemoller [99], 3.6 page 31 - 32). The existence of a G-map $E(F_S) \longrightarrow S(V_\infty)$ is seen as follows: If

$a : G/H \longrightarrow S(V)$ is a G-map, then

$$(u_1 g_1 H,\ u_2 g_2 H, \ldots) \longmapsto \sum_{j=1}^{\infty} u_j \cdot a(g_j H)$$

is a G-map from $G/H * G/H * \ldots$ into $\sum_{j=1}^{\infty} V - \{o\}$.

We have seen in 7.1 that localization allows to cut out suitable pieces of G-spaces. This is also true in the context of families. Let F be a family and X a G-space. Put

$$X_F = \{x \in X \mid G_x \in F\} \quad , \quad X^F = X \setminus X_F \ .$$

We assume that X, X_F etc. are numerable and that the pairs $(X, X_F), (A, A_F)$ have suitable excision properties.

<u>Proposition 7.4.2.</u> <u>The inclusion</u> $(X^F, A^F) \longrightarrow (X, A)$ <u>induces an isomorphism</u>

$$h_*[F_\infty, F]\ (X^F, A^F) \cong h_*[F_\infty, F]\ (X, A).$$

<u>Proof.</u> 7.2.3 gives $h\ (E(F) * X_F, X_F) = 0$. Since $E(F_\infty) = CE(F)$ we have as in the proof of 7.3.4 $h_*(E(F) * X_F, X_F) \cong h_*(E(F_\infty) \times X_F,\ E(F) \times X_F)$ and the latter group is by excision isomorphic to $h_*((E(F_\infty),\ E(F) \times (X, X^F))$. (One has to assume that this excision is actually possible.) The exact homology sequence of $h_*[F_\infty, F]$ for the pair (X, X^F) now yields the asserted isomorphism.

We have to discuss the excision problem. To begin with we have $h_*[F_\infty, F]\ (K) = 0$ for G-subsets K of K_F. If X is completely regular then X^F is closed in X (Palais [124] , 1.7.22). If $K \subset X_F$ is closed in X, then we have ordinary excision $h_*[F_\infty, F]\ (X\ K) = h_*[F_\infty, F]\ (X)$. In order to pass from $(X \setminus K)$ to $X \setminus X_F$ we must investigate the natural map

$$l : h_*[F_\infty, F]\,(X^F) \longrightarrow \text{inv lim}\; h_*[F_\infty, F]\,(U),$$

where the inverse limit is taken over the open G-neighbourhoods U of X^F, and see under which conditions l is an isomorphism.

Now one can use continuity conditions of the theory h_*. But for many spaces X one does not use this continuity. One notes that the inverse limit is taken over isomorphisms. Therefore l is injective if X^F is a G-retract of a neighbourhood U and l is surjective if a retraction $r : U \longrightarrow X^F$ is G-homotopic to the inclusion $U \subset X$.

We now discuss localization of equivariant homology at prime ideals of the Burnside ring and its relation to families of isotropy groups. Again we adopt an axiomatic approach.

We are given the G-equivariant theory $t_*^G(X,Y)$. We assume that $t_*^G(X,Y)$ is naturally a module over $A(G)$. We put $t_*^U(X,Y) = t_*^G(G/U \times X, G/U \times Y)$ and assume that t_*^U is an $A(U)$-module. The restriction

$$\text{res} = r : t_*^G(X) \longrightarrow t_*^U(X)$$

shall be compatible with the restriction $s : A(G) \longrightarrow A(U)$ i. e. $r(x \cdot y) = s(x) \cdot r(y)$, $x \in A(G)$, $y \in t_*^G(X)$. Moreover we have natural transformations (induction) ind : $t_*^U(U/K \times X) \longrightarrow t_*^U(X)$ such that the composion

$$t_*^U(X) \xrightarrow[\text{res}]{} t_*^U(U/K \times X) \xrightarrow[\text{ind}]{} t_*^U(X)$$

is multiplication with $U/K \in A(U)$.

Consider the prime ideal $q = q(H,p)$ of $A(G)$ (see 5. 7.) where $H < G$, NH/H is finite of order prime to p if $p \neq 0$. Assume that we have families $F_1 \supset F_2$ such that for $K \in F_1 \setminus F_2$ $q(K,p) = q(H,p)$. Let an index (p) or q denote localization at the prime ideal $(p) \subset Z$ or $q \subset A(G)$. Then we have

Proposition 7.4.3. *Multiplication with* $y \in q(H,p)$, *e. g.* $y = [G/H]$, *is an automorphism of the homology theory* $t_*^G [F_1, F_2]_{(p)}$. *The canonical map* $t_*^G [F_1,F_2]_{(p)} \longrightarrow t_*^G [F_1,F_2]_q$ *is an isomorphism.*

Proof. Using exact sequences 7.3.2 and the exactness of localization we see that it suffices to consider adjacent $F_1 \supset F_2$, say with $F_1 \setminus F_2 = (K)$ and $q(K,p) = q(H,p)$. We then use the isomorphism of 7.3.4. We abbreviate $NK = N$. The space $E(N/K)$ is the classifying space (in the sense of Segal [144]) of the category with objects the elements of N/K and exactly one morphism between any two objects. This category defines a simplicial space and its geometric realisation is $E(N/K)$. The skeleton filtration of this simplicial space gives a spectral sequence which has as E_2-term the homology of the following chain complex

$$\cdots \longleftarrow t_*(G \times_N (N/K^i \times Z) \xleftarrow[d_i]{} t_*(G \times_N (N/K)^{i+1} \times Z) \longleftarrow \cdots$$

with $Z = (CEF_2, EF_2) \times (X,A)$ and $d_i = \sum_{j=o}^{i} (-1)^j (pr_j)_*$ where pr_j omits the (1+j)-th factor. Multiplication by y, being a natural transformation of homology theories, induces an endomorphism of this spectral sequences. Hence it suffices to show that multiplication with y is an isomorphism on $t_*^G(G \times_N (N/K)^i \times (CEF_2, EF_2) \times (X,A))_{(p)}$ for $i \geq 1$. The group in question is isomorphic to $t_*^G(G/K \times (N/K)^{i-1} \times (CEF_2, EF_2) \times (X,A))_{(p)}$ and therefore the action of $y \in A(G)$ only depends on its restriction $y' \in A(K)$. By 5. 5. this restriction has the form

$y' = \varphi_K y\,[K/K] + \sum a_i\,[K/K_i]$ with $a_i \in Z$ and $(K_i) < (K)$, $(K_i) \neq (K)$. But $\varphi_K(y) \equiv \varphi_H(y) \not\equiv 0 \mod p$, because $y \in q(H,p)$. Since we localized at (p) multiplication with $\varphi_K(y)\,[K/K]$ is an isomorphism. The proof of 7.3.4 will be finished if we can show that multiplication with $[K/K_i]$ is zero. But by the axiomatic assumption this multiplication factors over $t_*^G(G/K \times K/K_i \times (N/K)^{i-1} \times (CEF_2, EF_2) \times (X,A))_{(p)}$ and this group is zero by 7.4.2.

7.5. Localization and splitting of equivariant homology.

Again we are given an equivariant homology theory t_*^G which is a module over A(G) such that the axioms of the previous section are satisfied. If we localize at (p) the theory becomes a module over $A(G)_{(p)}$. The idempotents of $A(G)_{(p)}$ split off direct factors we are going to describe these direct factors.

Let $q = q(H,p)$ a prime ideal of A(G) where H is the defining group of q (i. e. G/H q). We consider two chain complexes

$$t_*^G \xleftarrow{d_o} t_*^G(G/H) \xleftarrow{d_1} t_*^G((G/H)^2) \xleftarrow{d_2} \cdots$$

$$t_*^G \xrightarrow{d^o} t_*^G(G/H) \xrightarrow{d^1} t_*^G((G/H)^2) \xrightarrow{d^2} \cdots$$

with $d_i = \sum_{j=o}^{i} (-1)^j (pr_j)_*$ and $d^i = \sum_{j=o}^{i} (-1)^j (pr_j)^*$

(Here $(pr_j)^*$ is the induction (alias transfer) which is assumed to exist with suitable properties.)

<u>Proposition 7.5.1. The homology of these chain complexes is zero when localized at</u> q.

<u>Proof.</u> We define a contracting homotopy s for the first chain complex

by the formula

$$s = [G/H]^{-1}(pr_o)^* \ : \ t_*^G((G/H)^i)_q \longrightarrow t_*^G((G/H^{i+1})_q .$$

One verifies $ds + sd = id$ using that $pr_* \ pr^*$ is multiplication by $[G/H]$. A similar proof works for the second chain complex. (Compare also section 6.)

We apply the foregoing in the following situation. We put $t_*^G(G/H \times X) = t_*^H(X)$ for G-spaces X. The restriction $t_*^G(X) \longrightarrow t_*^H(X)$ becomes injective when localized at $q(H,p)$ and the image is equal to the kernel of

$$pr_o^* - pr_1^* \ : \ t_*^G(G/H \times X)_q \longrightarrow t_*^G(G/H^2 \times X)_q .$$

We denote this kernel by $t_*^H(X)_q^{inv}$, the invariant elements.

Let FH be the family of all subgroups subconjugate to H and let F'H be the family of those $K \in FH$ with $q(K,p) \neq q(H,p)$. Then we have a natural transformation of homology theories

(7.5.2) $$r_H \ : \ t_*^G(X)_{(p)} \longrightarrow t_*^H(X)_{(p)}^{inv} \longrightarrow t_*^H[FH,F'H](X)_{(p)}^{inv}$$

where the first map is restriction and the second comes from the exact homology sequence of the pair FH,F'H. (Note that EFH is H-contractible by 7.2.4)

Theorem 7.5.3. (a) $(r_H)_q$ is an isomorphism.

(b) r_H is split surjective.

(c) The product of the maps r_H

$$r = (r_H) \ : \ t_*^G(X)_{(p)} \longrightarrow \prod_{(H) \in \phi(p)} t_*^H[FH,F'H](X)_{(p)}^{inv}$$

<u>is injective and an isomorphism if only a finite number of factors on the right are non-zero</u>.

<u>Proof.</u> (a) From 7.4.3 we know that

$$t^H_* \,[FH, F'H]\, (X)^{inv}_{(p)} \cong t^H_* \,[FH, F'H]\, (X)^{inv}_{(p)}$$

because the isomorphism holds without "inv" and localization is exact. We have for any space X the isomorphism $t^G_*(X)_q = t^H_*(X)^{inv}_q$. What remains to be shown is that $t^H_*(X)_q \longrightarrow t^H_*[FH, F'H](X)_q$ is an isomorphism or, equivalently, that $t^H_*[F'H](X)_q$ is zero. Because of the additivity of the theory it is enough to show that $t^H_*(G/K \times X)_q = 0$ for $K \in F'H$. This follows from the homology version of 7.1.3 because $A(K)_{q(H,p)} = 0$.

(b) In view of (a) r_H is up to isomorphism obtained from tensoring the canonical map $A(G)_{(p)} \longrightarrow A(G)_q$ with $t^G_*(X)$. This canonical map is split surjective, because q has an associated idempotent $e(q) \in A(G)_{(p)}$ and $e(q)\, A(G)_{(p)} = A(G)_q$.

(c) The analogous assertion is true if we localize at maximal ideals of A(G).

<u>Remark 7.5.3.</u> Let G be a finite group. Let p be a prime number or 0. Write $|G| = p^k m$ with m prime to p if $p \neq 0$. Write $|G| = m$ is case p=0. If we can divide by m in the groups $t^G_*(X,A)$ then the map r in 7.5.3 is an isomorphism without localization at (p). In particular if we invert the order of the group, then the homology theory splits into summands

$$t^H_* \,[FH, F'H]\, (X)^{NH/H}$$

where FH (resp. F'H) is the family of all (resp. all proper) subgroups of G and the NH/H means the ordinary invariants under the NH/H-action.

Remark 7.5.4. We have seen that A(G) may contain many idempotents even without localization. Such idempotents split off direct factors from equivariant homology theories and these direct factors may be described using families. This is quite analogous to the considerations above. For details see tom Dieck [66] .

7.6. Transfer and Mackey structure.

We have to describe examples of homology theories which satisfy the axioms of 7.4. We use some homotopy theory which is developed in the next chapter which should be consulted for notation and some details. The application of the Burnside ring to equivariant (co-)homology and (co-) homotopy makes use of the Lefschetz fixed point index and fixed point transfer developed by Dold [76] , [77] in the non-equivariant case. We refer to these papers for details and further information. We recall the results that we need in a slightly different set up.

Let G be a compact Lie group. A G-map $p : E \longrightarrow B$ is called G-ENR_B (= euclidean G-neighbourhood retract over B) if there exists a real G-module V with G-invariant inner product, an open G-subset $U \subset B \times V$, and G-maps $i : E \longrightarrow U$, $r : U \longrightarrow E$ over B with $ri = id(E)$. Let $(B \times V)^c$ be the Thom space of the trivial bundle $B \times V \longrightarrow B$. Note that $(B \times V)^c$ is canonically G-homeomorphic to the smashed product $B^+ \wedge V^c$ where B^+ is B with a separate base point added.

If p,i, and r are as above, if p is a proper map and B locally compact and paracompact there exists a G-invariant continous function $\varrho : B \longrightarrow]0, \infty[$ such that for all $b \in B$ we have $\varrho(b) < d(ip^{-1})b)$, $\{b\} \times V \setminus U)$, where d denotes the metric derived from the inner product on V.

For such maps we call transfer map associated to the data p,i, and r any pointed G-map

$$h : (B \times V)^c \longrightarrow (E \times V)^c$$

with the following properties

(7.6.1) The inverse image of $E \times \{o\}$ under h is iE.

(7.6.2) For $u = (b,v) \in U$ and $2d(v, pr_2 iru) < \rho(b)$ the map h has the form
$h(u) = (ru,\ v - pr_2\ iru)$.

If X and Y are pointed G-spaces we let $\omega_G^o(X;Y)$ denote the direct limit over pointed G-homotopy sets $[V^c \wedge X,\ V^c \wedge Y]_G^o$ using suspensions over all (complex) G-modules; see chapter 8. Using suspension isomorphisms we extend this functor to functors $\omega_G^\alpha\ (X;Y)$, graded over α in the real representation ring RO(G) of G. We get a cohomology theory in the variable X and a homology theory in the variable Y.

Proposition 7.6.3. *Let* $p : E \longrightarrow B$ *be* G-ENR$_B$ *with retract representation* i,r *as above. Let* p *be proper and* B *locally compact and paracompact. Then transfer maps* h *exist and their pointed* G-*homotopy class is uniquely determined by* 7.6.1 *and* 7.6.2. *The stable* $\tilde{p} \in \omega_G^o(B^+;E^+)$ *of* h *is independent of the retract representation* i,r.

Proof. A proof may be extracted from Dold [77] . (Note that we consider a somewhat simpler situation.)

Example 7.6.4. Let $p : E \longrightarrow B$ be a submersion between compact differentiable G-manifolds. Let $j : E \longrightarrow V$ be an equivariant embedding into a G-module V. Then $i = (p,j) : E \longrightarrow B \times V$ is an embedding over B. A retract representation may be obtained from a tubular neighbourhood U of i. Hence p is G-ENR$_B$.

If $t_G^*(-)$ is a cohomology theory for G-spaces which has suspension isomorphisms for all G-modules (or all complex G-modules, etc.) then a transfer map h or $\tilde{p}$ as is in 7.6.3 induces a homomorphism

$$(7.6.5) \qquad p_! : t_G^*(E) \longrightarrow t_G^*(B)$$

called <u>transfer</u>. Similarly for homology theories t_*^G we get a transfer

$$(7.6.6) \qquad p^! : t_*^G(B) \longrightarrow t_*^G(E) .$$

The composition $p_! \, p^*$ is in the case of a multiplicative cohomology theory multiplication with the Lefschetz-Dold index $I_p \in t_G^0(B)$ (see Dold [76]). In particular we have the index $I(X) \in \omega_G^0$ for the map $X \longrightarrow$ Point, where X is a compact G-ENR and $\omega_G^0 = \operatorname{colim} [V^c, V^c]_G^0$ are the coefficients of equivariant stable cohomology in dimension zero. As usual ω_G^0 is a commutative ring with unit. In the next chapter we shall prove the following basic result.

<u>Theorem 7.6.7. The assignment induces a map</u> $I_G : A(G) \longrightarrow \omega_G^0$. <u>This map is an isomorphism of rings.</u>

We now collect the formal properties of the transfer which are used to establish the axioms used in the localization theorems in 7.4 and 7.5. We call a G-ENR$_B$ $p : E \longrightarrow B$ with p proper and B locally compact and paracompact a <u>transfer situation</u>. If P is a point we abbreviate $\omega_G^0(B;P^+) = \omega_G^0(B)$; this is a commutative ring, with unit if $B = C^+$. The cohomology group $t_G^*(B^+ \wedge X)$ carries a $\omega_G^0(B^+)$-module structure which is natural in X. The definition runs as follows: If $a \in \omega^0_G(B^+)$ is represented by $a : V^c \wedge B^+ \longrightarrow V^c$ let $a_1 : V^c \wedge B^+ \longrightarrow V^c \wedge B^+$ be given as $(v,b) \longmapsto (a(v,b),b)$. Then the action of a is the map

$$t^*_G(B^+ \wedge X) \cong t^*_G(V^c \wedge B^+ \wedge X) \xrightarrow[(a_1 \wedge id)^*]{} t^*_G(V^c \wedge B^+ \wedge X) \cong t^*_G(B^+ \quad X)$$

where the isomorphisms are suspensions. Similarly for homology. The next proposition collects what we need about the transfer and this module structure.

Proposition 7.6.8. Let $h : E' \longrightarrow E$ and $f : E \longrightarrow B$ be transfer situations.

(a) fh is a transfer situation and $h^! f^! = (fh)^!$, $f_! h_! = (fh)_!$.

(b) Let

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \phi\ } & E \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ B_1 & \xrightarrow{\ \varphi\ } & B \end{array}$$

be a pull-back and B_1 locally compact and paracompact. Then f_1 is a transfer situation and

$$f^! \varphi_* = \phi_* f_1^!, \quad (f_1)_! \varphi^* = \phi^* f_! .$$

(c) For $f_* : t^G_*(E^+ \wedge X) \longrightarrow t^G_*(B^+ \wedge X)$ and $a \in \omega^0_G(B^+)$ we have

$$f_*(f^* a \cdot s) = a \cdot f_*(s) .$$

(d) For $f^* : t^*_G(B^+ \wedge X) \longrightarrow t_G(E^+ \wedge X)$ and $b \in \omega^0_G(B^+)$ we have

$$f^*(b \cdot x) = f^*(b) \cdot f^*(x) .$$

(e) For $f^! : t^G_*(B^+ \wedge X) \longrightarrow t^G(E^+ \wedge X)$ and $a \in \omega^0_G(B^+)$ we have

$$f^!(a\cdot x) = f^*(a)\cdot f^!(x).$$

(f) For $f_! : t_G^*(E^+\wedge X) \longrightarrow t_G^*(B^+\wedge X)$ and $a \in \omega_G^o(B^+)$ we have

$$f_!(f^*a\cdot b) = a\cdot f_!(b).$$

(g) If $p : E \longrightarrow B$ is a transfer situation and $H < G$ a closed subgroup then the H-fixed point map $p^H : E^H \longrightarrow B^H$ is again a transfer situation (for the group NH/H) and $(p^H)^\sim = r\tilde{p}$, where $r : \omega_G^o(B^+;E^+) \longrightarrow \omega_{NH/H}^o(B^{H+};E^{H+})$ is induced by restriction to H-fixed points.

(h) If $p : E \longrightarrow B$ is a transfer situation for the subgroup H of G then $G \times_H p : G \times_H E \longrightarrow G \times_H B$ is a transfer situation for the group G and $j(\tilde{p}) = (G \times_H p)^\sim$ where

$$j : \omega_H^o(B^+;E^+) \longrightarrow \omega_G^o(G \times_H B^+;G \times_H E^+)$$

is induced by the functor $X \longmapsto G \times_H X$.

For the proof of (a) and (b) we refer to the above mentioned work of Dold. Using this and our description of transfer maps, (c) to (h) become fairly routine verifications.

The applications to the axiomatic treatment in 7.4 is as follows: $res : t_*^G(X) \longrightarrow t_*^G(G/H \times X)$ is the transfer for $f : G/H \longrightarrow$ Point and $ind : t_*^G(G/H \times X) \longrightarrow t_*^G(X)$ is induced by f. The relevant properties follow from 7.6.7 and 7.6.8.

For finite groups there exist important equivariant homology theories which are not stable in the sense that they admit suspension isomorphisms for enough G-modules. Examples are the bordism theories of

Conner and Floyd. Nevertheless the methods of 7.4 and 7.5 are applicable. The relevant axioms can be established by direct geometric methods, without using transfer and stable homotopy as above. For bordism theories "restriction" is just the usual restriction to a subgroup and "induction" is induced by the functor $X \longmapsto G \times_H X$ from H-spaces to G-spaces. For an axiomatic treatment along these lines see tom Dieck [60] . The Bredon equivariant homology and cohomology (Bredon [36] , Bröcker [38] , Illman) have canonical restriction and induction if the coefficient system is a Mackey functor.

7.7. Localization of equivariant K-theory.

In order to add some meat to the vegetable soup 7.1 - 7.6 we consider equivariant K-theory as an example of the previous general theory. Of course, one can treat K-theory more directly, using representation theoretic methods. We let $K_G(X)$ be the Grothendieck ring of complex G-vector bundles over the (compact) G-space X (see Segal [142]).

Let G be a compact Lie group. As in Segal [143] we use the

Definition 7.7.1. A closed subgroup S of G is called Cartan subgroup of G if NS/S is finite and S is topologically cyclic (i. e. powers of a suitable elements are dense $\Leftrightarrow$ S is the product of a torus and a finite cyclic group). A Cartan subgroup is p-regular if the group of components has order prime to p, for a prime number p.

Let C be the set of conjugacy classes of Cartan subgroups of G and C(p) the subset of p-regular groups. We refer to Segal [143] for the proof of

Proposition 7.7.2. The set C is finite.

If $(S) \in C(p)$, $P < NS/S$ a p-Sylow subgroup and $Q < NS$ the pre-image of P then $|NQ/Q| \not\equiv 0 \bmod p$. Hence $Q = Q_S$ is the defining group of the prime ideal $q(S,p)$.

By the equivariant Bott-isomorphism the cohomology theory $K_G(-)$ has suspension isomorphism for complex G-modules. Thus $K_G(-)$ becomes and A(G)-module and $K_G(\text{Point}) = R(G)$ becomes an A(G)-algebra. Actually the map $A(G) \longrightarrow R(G)$ which comes from the homotopy considerations of 7.6. coincides with the equivariant Euler characteristic of chapter 5.

If $H < G$ let H_p be the smallest normal subgroup such that H/H_p is a p-group.

Proposition 7.7.3. $R(G)_{q(H,p)} = 0$ *if and only if* H_p *is a* p-*regular Cartan subgroup*.

Proof. Let $S < G$ be a topologically cyclic subgroup with generator G. The diagram

$$\begin{array}{ccc} A(G) & \xrightarrow{\chi_G} & R(G) \\ \varphi_S \downarrow & & \downarrow e_g \\ Z & \xrightarrow{\subset} & \mathbb{C} \end{array}$$

is a commutative diagram of ring homomorphisms (χ_G equivariant Euler characteristic 5. 5. 6 ; e_g evaluation of characters at g). We view everything as A(G)-module and localize at $q = q(H,p)$. Since elements of R(G) are detected by the various e_g we can find an S with $\mathbb{C}_q \neq 0$ if $R(G)_q \neq 0$. But then $Z_q \neq 0$ and this implies $q(S,p) = q(H,p)$. Since S is cyclic there exists a Cartan subgroup T with $S \triangleleft T$ such that T/S is torus, by Segal [143] , 1.2 and 1.5. Hence $q(T,p) = q(S,p)$. One can take a p-regular subgroup T' of T with $q(T,p) = q(T',p)$. The assertion then follows from 5. . An analogous argument shows that $R(G)_{q(S,p)} \neq 0$ for a p-regular Cartan group p.

From 7.7.3 and 7.6 we obtain natural isomorphisms

(7.7.4) $$K_G(X)_{(p)} \cong \bigoplus_{(S) \in C(p)} K_G(X)_{q(S,p)} \qquad p \neq 0$$

(7.7.5) $$K_G(X)_{(0)} \cong \bigoplus_{(S) \in C} K_G(X)_{q(S,0)}$$

(7.7.6) $$K_G(X)_{q(S,p)} \cong K_{QS}(X)^{inv}_{q(S,p)}$$

where $QS < NS$ is the pre-image of a p-Sylow subgroup of NS/S. Moreover in 7.7.6 X can be replaced by $X(S) = \{ x \mid q(G_{x,p}) = q(S,p) \}$.

We are going to study the case of finite groups G more closely. Then S is a cyclic group of order prime to p and we have $1 \longrightarrow S \longrightarrow QS = H \longrightarrow P \longrightarrow 1$ with a p-group P, hence H is a semi-direct product and a p-hyperelementary group. Moreover

$$K_H(X)_{q(H,p)} = K_H(X^S)_{q(H,p)} .$$

One can describe H-equivariant vector bundles over X^S. The fibre consists of S-modules and these have to be grouped together according to the conjugation action of P.

We specialize further to the case $H = S \times P$. Then naturally $K_H(X^S) = R(S) \otimes K_p(X^S)$. Moreover $A(H) = A(S) \otimes A(P)$ and the following diagram of equivariant Euler characteristics is commutative

$$\begin{array}{ccccc} A(S) & \longrightarrow & A(H) & \longleftarrow & A(P) \\ \downarrow \chi_S & & \downarrow \chi_H & & \downarrow \chi_P \\ R(S) & \longrightarrow & R(H) & \longleftarrow & R(P) \end{array}$$

Let S be the cyclic group of order m and generator g. Suppose $(m,p) = 1$. Let x denote the irreducible standard representation of G. Then $R(S) \cong Z[x]/(x^m-1)$. Let $E = \{1-x^i \mid 1 \leq i \leq m-1\}$ be the set of Euler classes of non-trivial irreducible S-modules. Let $e : R(S) \longrightarrow Z[u_m]$ be evaluation of characters at g; here u_m is a primitive m-th root of unity.

<u>Proposition 7.7.7. The map</u> e <u>induces an isomorphism of rings</u>

$$\tilde{e} : R(S)[E^{-1}] \cong Z[m^{-1}, u_m].$$

<u>Proof.</u> We have to invert the $1 - u_m^i$, $1 \leq i \leq m-1$. If $m = p_1^{a(1)} \ldots p_r^{a(r)}$ is the factorization into prime powers and if u(i) is a primitive $p_i^{a(i)}$-th root of unity then 1-u(i) has norm p_i hence is invertible in $Z[m^{-1}, u_m]$. Moreover we see that m^{-1} and u_m are in the image of e. Therefore e is surjective. The map e factorizes .

$$Z[x]/(x^m-1) \xrightarrow[e_1]{} Z[x]/\Phi_m(x) \xrightarrow[e_2]{} Z[u_m]$$

where Φ_m is the m-th cyclotomic polynomial. The map e_2 is an isomorphism. If we put $x^m-1 = \Phi_m(x)\, P_m(x)$ then Φ_m and P_m are relatively prime and the canonical map

$$Z[x]/(x^m-1) \longrightarrow Z[x]/\phi_m \oplus Z[x]/P_m$$

is injective. The prime factors of P_m divide certain $1-x^i$, $1 \leq i \leq m-1$, and since these elements are to be inverted the P_m have to be inverted too. This can only happen if the localization E^{-1} trivialises the factor $Z[x]/P_m$, so that

$$Z[x]/(x^m-1)[E^{-1}] \longrightarrow Z[x]/\phi_m \; E^{-1}$$

must be injective and hence $\check{e}$ is injective too.

<u>Proposition 7.7.8. The map</u> e <u>induces an isomorphism of rings</u>

$$e' : R(S)_{q(S,p)} \longrightarrow Z_{(p)}[u_m] .$$

<u>Proof.</u> We have to invert the image of $A(S) \setminus q(S,p)$ under $\chi_S : A(S) \longrightarrow R(S)$. If $y \notin q(S,p)$ then $e\chi_S(y) = |y^g| = |y^S| \not\equiv 0(p)$. Hence e induces a surjective map e'. The product of the Euler classes $\prod_{i=1}^{m-1} (1-x^i)$ is a rational representation and therefore equal to $\chi_S(y)$ for a suitable $y \in A(S)$. One has $|y^S| = m$, so $y \notin q(S,p)$. Hence the map in question is a localization of e in 7.7.7 and therefore injective.

We now come back to $H = S \times P$. We note that $A(P)_{q(P,p)} = A(P)_{(p)}$ is a local ring and

$$A(H)_{q(H,p)} \cong A(S)_{q(S,p)} \otimes A(P)_{q(P,p)}$$

and more generally therefore

$$(7.7.9) \qquad K_H(X^S)_{q(H,p)} \cong R(S)_{q(S,p)} \otimes K_p(X^S)_{(p)} .$$

Corollary 7.7.10. Let $m = |G|$. Then we have a canonical isomorphism of rings

$$K_G(X)\,[m^{-1}] \cong \bigoplus_{(C)} (R(C)\,[E_C^{-1}] \otimes K(X^C))^{NC/C}$$

where (C) runs through the conjugacy classes of cyclic subgroups of G, and $E_C \subset R(C)$ is the set of Euler classes of non-trivial irreducible C-modules.

7.8. Localization of the Burnside ring.

Let $F_1 \supset F_2$ be families of subgroups of G. We denote by $A(G;F_1)$ the ideal of A(G) generated by sets (or spaces) X with isotropy groups in F_1 and by $A(G;F_1,F_2)$ the ideal $A(G;F_1)$ modulo the subideal $A(G;F_2)$.

For simplicity let G be a finite group. If $(H) \in \phi(p)$, i. e. $|NH/H| \not\equiv 0$ mod p let H_p be the smallest normal subgroup such that H/H_p is a p-group. Then $\{K | q(K,p) = q(H,p)\} = \{K | (H_p) \lesssim (K) \leq (H)\}$. Call this set $F_o(H)$. We put $F(H) = \{K | (K) \leq (H)\}$ and $F'(H) = F(H) \setminus F_o(H)$.

The ring $A(G)_{(p)}$ splits into a direct product of rings $A(G)_{q(H,p)}$, $(H) \in \phi(p)$, and these factors may also be written as $e(H)\ A(G)_{(p)}$ where e(H) is a suitable indecomposable idempotent element of $A(G)_{(p)}$.

Proposition 7.8.1. Taking H_p-fixed points induces an isomorphism

$$A(H;FH,F'H) \cong A(H/H_p) \ .$$

Proof. Both groups have as an additive basis the H/K, $(H_p) \leq (K) \leq (H)$, and $H/K^{H_p} = H/K$.

Proposition 7.8.2. The following groups are canonically isomorphic

$$A(G)_{q(H,p)},\quad A(G;FH)_{q(H,p)},\quad A(G;FH,F'H)_{q(H,p)}$$

and $$A(G;FH,F'H)_{(p)}.$$

<u>Proof.</u> The quotient map $A(G;FH) \longrightarrow A(G;FH,F'H)$ becomes an isomorphism after localization at $q(H,p)$ because the kernel $A(G;F'H)$ is detected by fixed point mappings $\varphi_L : A(G;F'H) \longrightarrow Z$ with $q(L,p) \neq q(H,p)$ and therefore $Z^L_{q(H,p)} = 0$ where $Z^L = Z$ is an $A(G)$-module via φ_L. For a similar reason the inclusion $A(G;FH) \longrightarrow A(G)$ induces an isomorphism of its $q(H,p)$-localizations. The canonical map $A(G;F,F'H)_{(p)} \longrightarrow A(G;FH,F'H)_{q(H,p)}$ is an isomorphism by an argument and in the proof of 7.

The idempotent $e(H)$ is contained in $A(G;FH)_{(p)}$ and multiplication by $e(H)$ induces a split surjection $A(G)_{(p)} \longrightarrow A(G;FH,F'H)_{(p)}$ which corresponds to the canonical map $A(G)_{(p)} \longrightarrow A(G)_{q(H,p)}$ under the isomorphisms of 7.8.2. By the general theory we have an isomorphism

$$(7.8.3) \qquad A(G;FH,F'H)_{(p)} = A(H;FH,F'H)^{inv}_{(p)} .$$

Combining with 7.8.1 we obtain

<u>Proposition 7.8.4. Taking</u> H_p-<u>fixed points for the various</u> $(H) \in \phi(p)$ <u>induces a ring isomorphism</u>

$$A(G)_{(p)} \longrightarrow \prod_{(H) \in \phi(p)} A(H/H_p)^{inv}_{(p)}$$

<u>and the corresponding map into the product without "inv" is a split monomorphism of rings</u>.

7.9. Comments.

For localization of equivariant K-theory see Atiyah-Segal and Segal [142] ; for equivariant cohomology: Quillen [127] , Hsiang ; for bordism theory tom Dieck [53] , [58] , [59] Wilson [167] ; for cohomotopy and general theory: Kosniowski [105] , tom Dieck [56] , [57] , [60] . The presentation in this section is mainly drawn from the author's papers and unpublished manuscripts.

8. Equivariant Homotopy Theory

8.1. Generalities.

Let G be a compact Lie group. We consider various categories obtainable from G-spaces:

- G-Top : The category of G-spaces and G-maps.
- $G\text{-Top}^{o}$: The category of G-spaces with base point o (always fixed under G) and base-point preserving G-maps.
- G-Top (2): Pairs (X,A) of G-spaces and G-maps of pairs.
- $G\text{-Top}^{o}(2)$: Pairs of pointed G-spaces.

All these categories have their associated notion of homotopy. For sets of G-homotopy classes we use the following notation (resprectively):

$$[X,Y]_G \ , \ [X,Y]_G^{o} \ ,$$

$$[(X,A),\ (Y,A)]_G \ , \ [(X,A),\ (Y,A)]_G^{o} \ .$$

Usually we restrict to suitable subcategories, using notation that should be self-explanatory, e. g. G-CW for the category of G-CW complexes (to be defined later), $G\text{-CW}^{o}$, G-CW(2), $G\text{-CW}^{o}(2)$. The standard constructions of homotopy theory using the unit interval , like suspension, mapping cone, path space can be done in G-Top, $G\text{-Top}^{o}$, etc. using trivial G-action on $I = [0,1]$. There are resulting Barrat-Puppe sequences and their Eckmann-Hilton duals for fibrations. A G-cofibration $i : A \longrightarrow X$ should have the homotopy extension property in G-Top, a G-fibration $p : E \longrightarrow B$ should have the homotopy lifting in G-Top. Of course the problem remains to characterise G-cofibrations etc. in terms of other data, e. g. by considering fixed point sets. This is very important and we return to such questions from time to time (see e. g. the discussion of G-ENR's in I. 5.2). The general theme is to reduce equivariant problems to problems in ordinary topology and the general method will be: induction over the orbit types. For a single orbit type one often has a problem about ordinary bundles (e. g. existence of sections). A basic example of this procedure is the construction and classification of G-maps via sections of an auxiliary map. We describe this transition.

Let X and Y be G-spaces. For a G-map $f : X \longrightarrow Y$ we must have $G_x < G_{fx}$ for all $x \in X$. Therefore we consider the subspace

(8.1.1) $$I(X,Y) := \{(x,y) \mid G_x \subset G_y\} \subset X \times Y.$$

This is a G-subspace of X x Y with the diagonal action. Let (X;Y) be the orbit space. The projection X x Y induces

(8.1.2) $$q : (X;Y) \longrightarrow X/G.$$

The G-map $f : X \longrightarrow Y$ induces $X \longrightarrow I(X,Y) : x \longmapsto (x,fx)$ and by passing to orbit spaces we obtain a section $s_f : X/G \longrightarrow (X;Y)$ of q.

Proposition 8.1.3. The assignment $f \longmapsto s_f$ *induces a bijection between the set of G-maps* $X \longrightarrow Y$ *and the set of sections of* q. *Two G-maps* $f_1, f_2 : X \longrightarrow Y$ *are G-homotopic if and only if the corresponding sections are homotopic*.

Proof. We claim that

(8.1.4) $$\begin{array}{ccc} I(X,Y) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ (X;Y) & \underset{q}{\longrightarrow} & X/G \end{array}$$

is a pull-back diagram. Let $Z \longrightarrow (X;Y)$ be the pull-back of p along q. Since $I(X,Y) \longrightarrow X$ is isovariant we obtain from the commutative diagram 8.1.4 a G-map $I(X,Y) \longrightarrow Z$ over $(X;Y)$ which is bijective. In any pull-back diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & X/G \end{array}$$

the map q is canonically homeomorphic to the orbit map $Z \longrightarrow Z/G$. Since X and Y are assumed to be Hausdorff spaces the spaces $I(X,Y)$, Z and their orbit spaces are Hausdorff and the orbit maps are proper (Bourbaki [32], III § 4.1. Prop. 2). By Bourbaki [32], I § 10.1. Prop. 5 the map $I(X,Y) \longrightarrow Z$ is proper and therefore, being bijective, a homeomorphism.

Now given a section $s : X/G \longrightarrow (X;Y)$ we have in the pull-back 8.1.4 the induced section $t : X \longrightarrow I(X,Y)$ which composed with the projection $I(X,Y) \longrightarrow Y$ yields a G-map $f_s : X \longrightarrow Y$. (Verify that t is a G-map.) The correspondences $s \longmapsto f_s$, $f \longmapsto s_f$ are seen to be mutually inverse. A G-homotopy $X \times I \longrightarrow Y$ induces a section $(X \times I)/G \longrightarrow (X \times I;Y)$ which, via canonical homeomorphisms $(X \times I)/G \cong X/G \times I$ and $(X \times I;Y) \cong (X;Y) \times I$ corresponds to a homotopy of sections (and vice versa).

We now explain the principle of constructing G-maps via induction over orbit-types. Suppose that Or is a finite set of conjugacy classes of subgroups of G. We can choose an admissible indexing $Or = \{(H_1), (H_2), \dots, (H_k)\}$, this meaning that $(H_j) < (H_i)$ implies $i < j$. If the G-space X has finite orbit type we always choose an admissible indexing of its set of orbit types $Or(X)$. Let $f : X \longrightarrow Y$ be a G-map between spaces of finite orbit-type. Let

$$\mathrm{Or}(X) \cup \mathrm{Or}(Y) = \{(H_1),\dots,(H_k)\}$$

be an admissible ordering. Define a filtration of X by closed G-subspaces

$$X_1 \subset X_2 \subset \dots \subset X_k = X$$

$$X_i = \{x \in X \mid \text{for some } j \leq i \quad (G_x) = (H_j)\} .$$

Then $X_i \setminus X_{i-1}$ is the orbit bundle $X_{(H)}$, $H = H_i$. The G-map f induces G-maps $f_i : X_i \longrightarrow Y_i$. If a G-map $k : X_{i-1} \longrightarrow Y_{i-1}$ is given we are interested in its extensions $K : X_i \longrightarrow Y_i$.

Proposition 8.1.5. The extensions K of k are in bijective correspondence with the NH/H-extensions $e : X_i^H \longrightarrow Y_i^H$ of $k^H : X_{i-1}^H \longrightarrow Y_{i-1}^H$ $(H = H_i)$.

Proof. Given K we have $e = K^H$ and since $GX_H = X_i \setminus X_{i-1}$ the G-map K is uniquely determined by K^H. Now suppose we are given an NH/H-map $e : X_i^H \longrightarrow Y_i^H$ extending k^H. We define a map

$$\begin{aligned} &E : X_i \longrightarrow Y_i && \text{by} \\ &E(x) = K(x) && \text{if} \quad x \in X_{i-1} \\ &E(x) = g\,e(y) && \text{if} \quad x = gy,\ y \in X_i^H . \end{aligned}$$

We have to show that E is well-defined and continuous. If $x = g_1y_1 = g_2y_2$ and $y_1 \in X_{i-1}^H$ then $y_2 \in X_{i-1}^H$ and $g_1e(y_1) = g_1K(y_1) = K(g_1y_1) = K(x)$ $= g_2e(y_2)$ because K is a G-map. If $x = g_1y_1 = g_2y_2$ and $y_1,y_2 \in X_i^H \setminus X_{i-1}^H$ then $g_1 = g_2n$ with $n \in NH$ and therefore

$$g_1e(y_1) = g_2ne(y_1) = g_2e(ny_1) = g_2e(y_2)$$

because e is an NH-map. Hence E is well-defined. E is continuous on the closed subsets X_{i-1} and GX_1^H, hence continuous.

We combine 8.1.3 and 8.1.5 in the following manner: The action of NH/H on $X_i^H \setminus X_{i-1}^H$ is free. Hence we are in the following situation: Let (X,A) and (Y,B) be pairs of G-spaces (A and B closed subspaces). The action of G on $X \setminus A$ and $Y \setminus B$ shall be free. We want to extend G-maps $f : A \longrightarrow B$ to G-maps $F : X \longrightarrow Y$. By 8.1.3 we have to extend a partial section of $(X;Y) \longrightarrow X/G$ given over A/G (a closed subspace of X/G) to a section. But over $(X \setminus A)/G$ we have an ordinary fibre bundle with fibre Y (locally trivial by the slice theorem). (See Bredon [37], II. 2 for the special case of free actions.) So one usually encounters a sequence of fibre bundle problems and moreover one has to deal with the singular behaviour of $(X;Y) \longrightarrow X/G$ over A and near A.

8.2. Homotopy equivalences.

We show that under suitable hypotheses a G-map $f : X \longrightarrow Y$ is a G-homotopy equivalence if and only if the fixed point mappings f^H are ordinary homotopy equivalences. This holds in particular if X and Y are G-ENR's.

An assertion as above should be true if X and Y are free G-spaces. This is a fibre bundle problem. A free G-space X is called <u>numerable</u> if $X \longrightarrow X/G$ is a numerable principal G-bundle in the sense of Dold [71] , i. e. locally trivially over an open cover which has a subordinate locally finite partition of unity.

<u>Proposition 8.2.1. Let $f : X \longrightarrow Y$ be a G-map from a G-space to a numerable free G-space Y. Then f is a G-homotopy equivalence if and only if f is an ordinary homotopy equivalence.</u>

<u>Proof.</u> Certainly X must be a free G-space. Since X maps into a locally

trivial space if is itself locally trivial (Bredon [37] , II. 3.2). Moreover $X \longrightarrow X/G$ is numerable, by pulling back a numeration of $Y \longrightarrow Y/G$. Let $EG \longrightarrow BG$ be the universal principal G-bundle (this is numerable, Dold [71] , 8). Consider the following diagram of G-maps

$$\begin{array}{ccc} EG \times X & \xrightarrow[\text{id} \times f]{} & EG \times Y \\ \Big\downarrow \text{pr} & & \Big\downarrow \text{pr} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We show that pr and id x f are G-homotopy equivalences. The map (idxf)/G

$$\begin{array}{ccc} (EG \times X)/G & \longrightarrow & (EG \times Y)/G \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

is a fibre-wise map over BG between fibrations. The induced map on each fibre is an ordinary homotopy equivalence because f is. By Dold [71] , 6.3. and 8. (id x f)/G is a fibre homotopy equivalence and by the covering homotopy theorem for bundle maps Dold [71] , 7.8, the map id x f is a bundle equivalence hence a G-homotopy equivalence. A similar argument applies to pr: The map $(EG \times X)/G \longrightarrow X/G$ is a fibration with contractible fibre EG hence a homotopy equivalence (actually shrinkable, Dold [71] , 3.2). Now apply the covering homotopy theorem for bundle maps again.

<u>Proposition 8.2.2.</u> <u>Given a diagram of</u> G-<u>spaces and</u> G-<u>maps</u>

$$\begin{array}{ccc} Y & \xrightarrow{\quad p \quad} & Z \\ {\scriptstyle f_A}\uparrow & & \uparrow{\scriptstyle h} \\ A & \subset & X \end{array}$$

and a G-homotopy $H_A : h|A \simeq pf_A$. *Assume that* $A \subset X$ *is a G-cofibration. Then there exists a G-map* $f : X \longrightarrow Y$ *extending* f_A *and a G-homotopy* $H : h \simeq pf$ *extending* H_A *provided*

(a) *p is an equivariant homotopy equivalence*

or (b) *p is an ordinary homotopy equivalence and* $X \setminus A$ *is a numerable free G-space.*

Proof. Replace p by the equivariantly homotopy equivalent G-fibration $q : E \longrightarrow Z$, where E is the path-space

$$E = \{(w,y) \in Z^I \times Y \mid w(1) = p(y)\} \quad , \; q(w,y) = w(o).$$

The G-action on E is given by $g(w,y) = (g\cdot w, gy)$, where $(g\cdot w)(t) = gw(t)$. Let $r : F \longrightarrow X$ be the G-fibration over X induced by, i. e.

$$F = \{(x,w,y) \in X \times Z^I \times Y \mid w(o) = h(x),\; w(1) = p(y)\}$$

$$r(x,w,y) = x.$$

Define $k : A \longrightarrow F$ by $k(a) = (a, w_a, f_A(o))$ with

$$w_a(t) = \begin{cases} h(a) & o \leq t \leq 1/2 \\ H_A(a, 2t-1) & \frac{1}{2} \leq t \leq 1 \;. \end{cases}$$

Then k is an equivariant section of r over A. From the description of F above we see that the theorem is proved if we can extend k to an equivariant section of r over X.

Since $A \subset X$ is a G-cofibration, there is an equivariant map $u : X \longrightarrow I$ and a G-homotopy $K : X \times I \longrightarrow X$ such that $A \subset u^{-1}(o)$, $K(x,o) = x$, $K(a,t) = a$ for all $a \in A$ and $t \in I$, and $K(x,1) \in A$ for $x \in u^{-1}[0,1[$ (this is the equivariant analogue of Strøm [156]; see also tom Dieck-Kamps-Puppe [70], § 3). Put $U = u^{-1}[0,1[$. Extend k to an equivariant section r over U by $k(x) = (x, w_x, f_A K(x,1))$ with

$$w_x(t) = \begin{cases} hK(x,2t) & o \leq t \leq 1/2 \\ H_A(K(x,1),2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad x \in U$$

The restriction $r_{X \setminus A} : F_{X \setminus A} = r^{-1}(X \setminus A) \longrightarrow X \setminus A$ is G-shrinkable: Since p is a homotopy equivalence and a G-fibration it is shrinkable (Dold [71], 6.2), hence the induced r is shrinkable (Dold [71], 3.1). Hence $r_{X \setminus A}$ is a homotopy equivalence and by 8.2.1 (in case (b)) G-homotopy equivalence, and, being a G-fibration, $r_{X \setminus A}$ is shrinkable. (In case (a) $r_{X \setminus A}$ is induced from the G-shrinkable q). G-Shrinkable means: There exists an equivariant section t of $r_{X \setminus A}$ and a G-homotopy over $X \setminus A$ L from the identity to $tr_{X \setminus A}$. The required equivariant section s of r over X is now given by

$$s(x) = \begin{cases} t(x) & x \in X \setminus U \\ L(k(x), \max[2u(x)-1, o]) & x \in U \setminus A \\ k(x) & x \in A \end{cases}$$

<u>Proposition 8.2.3.</u> <u>Let</u> $p : (X,A) \longrightarrow (Y,B)$ <u>be a G-map such that</u> $p_A = p|A : A \longrightarrow B$ <u>is a G-homotopy equivalence and</u> p <u>is an ordinary homotopy equivalence. Suppose that</u> $X \setminus A$ <u>and</u> $Y \setminus B$ <u>are numerable free</u>

G-*spaces and* $A \subset X$, $B \subset Y$ *are* G-*cofibrations. Then any* G-*homotopy inverse* q_B *of* p_A *can be extended to a* G-*homotopy inverse* q *of* p *and any* G-*homotopy* $H_B : id_B \simeq p_A\, q_B$ *to a* G-*homotopy* $H : id_Y \simeq pq$.

Proof. We apply 8.2.2 (b) to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & Y \\ {\scriptstyle q_B}\uparrow & & \uparrow{\scriptstyle id} \\ B & \underset{i}{\subset} & Y \end{array}$$

and obtain a G-extension $q : Y \longrightarrow X$ of q_B and $H : Y \times I \longrightarrow Y$ of iH_B such that $H : id_Y \simeq pq$. Hence $(p,p_A)(q,q_B) \simeq id$ as maps between G-pairs. Since p was an ordinary homotopy equivalence q must be an ordinary homotopy equivalence. Hence we can apply 8.2.2 (b) once more to find an extension $\bar{p} : X \longrightarrow Y$ of p_A such that $(q,q_B)(\bar{p},p_A) \simeq id$ as maps of G-pairs. Hence (q,q_B) is a G-homotopy equivalence of G-pairs with G-homotopy inverse (p,p_A).

Proposition 8.2.4. Let $f : X \longrightarrow Y$ *be a* G-*map such that for all* $H < G$ *the map* f^H *is an ordinary homotopy equivalence. Suppose that for all* $H < G$ X_H, Y_H *are numerable free* NH/H-*spaces and* $G(X^H \setminus X_H) \subset GX^H$, $G(Y^H \setminus Y_H) \subset GY^H$ *are* G-*cofibrations. Suppose moreover that* X *and* Y *have finite orbit-type. Then* f *is a* G-*homotopy equivalence.*

Proof. Choose an admissible indexing of $Or(X) \cup Or(Y)$ as explained in 8.1. We have the associated filtration (X_n) and (Y_n) of X and Y and we show by induction over n that $f_n : X_n \longrightarrow Y_n$ is a G-homotopy equivalence. The induction starts, using 8.2.1. Suppose f_{n-1} is a G-homotopy equivalence with inverse h_{n-1}. Using 8.2.3 we see that h^H_{n-1} can be extended to an NH/H-homotopy inverse of f_n^H if $X_n \setminus X_{n-1} = X_{(H)}$. By 8.1.5 we find the required of f_n.

Remark 8.2.5. The hypotheses of 8.2.4 are satisfied if X and Y are G-ENR's. This follows from the theorem of Jaworowski 5.2.6 and the fact that an inclusion of G-ENR's is a G-cofibration.

We also mention a theorem of Segal-James [101], Theorem 1.1, giving another variant of 8.2.4.

Proposition 8.2.6. *Let X and Y be G-ANR's. Then a G-map $f : X \longrightarrow Y$ is a G-homotopy equivalence if the map $f^H : X^H \longrightarrow Y^H$ is a homotopy equivalence for all closed subgroups H of G.*

8.3. Obstruction theory.

According to 8.1.5 the basic extension problem in equivariant homotopy theory may be formulated as follows:

Extension problem: Given G-spaces $A \subset X$, A closed in X, and Y and a G-map $f : A \longrightarrow Y$. Suppose G acts freely on $X \setminus A$. Can f be extended to a G-map $F : X \longrightarrow Y$? If F exists, how can one classify G-homotopy classes of such extensions?

We want to reduce these problems to problems in classical obstruction theory, as presented in the books by Steenrod [154] or Baues [17] . By 8.1.3 we have to consider $q : (X;Y) \longrightarrow X/G$ with given partial section $s : A/G \longrightarrow (X\ Y)$ corresponding to f and we have to extend this section over X/G. This looks like a problem in obstruction theory, but the additional technical problem that arises comes from the fact that q is not, in general, a fibration. Over $(X \setminus A)/G$, q is the fibre bundle $((X \setminus A) \times Y)/G \longrightarrow (X \setminus A)/G$ with fibre Y, but when we approach

A/G the fibre change (the fibration has "singularities"). One possibility to circumvent this problem is to assume that the section s has an extension to a neighbourhood, i. e. the G-map f may be extended to a neighbourhood. This is the case when $A \subset X$ is a G-retract of a neighbourhood an in particular when $A \subset X$ is a G-cofibration, or when Y is a G-ANR and X is normal. (This extension property is the definition of a G ANR in Palais [124], 1.6. In particular a G-ENR is a G-ANR.)

<u>Proposition 8.3.1. Let</u> (X,A) <u>be a relative</u> G-CW-<u>complex of dimension</u> $\leq n$ <u>with free</u> G-<u>action on</u> $X \setminus A$. <u>Let</u> Y <u>be a</u> G-<u>space which is</u> n-<u>connected and</u> n-<u>simple</u> $(n \geq 1)$. <u>Then any</u> G-<u>map</u> $f : A \longrightarrow Y$ <u>has an extension</u> $F: X \to Y$. <u>The</u> G-<u>homotopy classes</u> rel. A <u>of such extensions correspond bijectively to elements of</u> $H^n(X/G, A/G; \pi_n Y)$ (where singular cohomology with suitable local coefficients is used).

<u>Remarks.</u> The assumption about (X,A) means that X is obtained from A by attaching cells $G \times D^i$ for $i \leq n$. Then (X/G,A/G) is an ordinary relative CW-complex of dimension $\leq n$. The inclusion $A \subset X$ is a G-cofibration, in fact a strong neighbourhood deformation retract (in G-Top): There exists a G-neighbourhood U of A in X such that $A \subset U$ is a G-homotopy equivalence rel. A. Over $X \setminus A$ we have the local coefficient system $((X \setminus A) \times \pi_n Y)/G \longrightarrow (X \setminus A)/G$ where the G-action on Y induces an action on $\pi_n Y$. By excision
$H^n(X/G, A/G; \pi_n Y) \cong H^n(X \setminus A/G, U \setminus A/G; \pi_n Y)$ and in the latter group we use the local coefficient system just defined.

<u>Proof.</u> Using 8.1 the problem is translated into a section extension problem and then classical obstruction theory is applied.

One of the immediate applications of obstruction theory is a proof of H. Hopf's theorem which determines the homotopy classes of maps from an n-manifold into an n-sphere. We generalize this to the equivariant situation in the next section.

8.4. The equivariant Hopf theorem.

A classical theorem of H. Hopf asserts that the homotopy classes of a closed connected orientable n-manifold M into the n-sphere are characterized by their degree and every integer occurs as degree of a suitable map. If M and S^n carry free actions of a finite group G then the equivariant homotopy classes are still determined by their degree, but no longer does every integer occur as a degree (e. g. if $G = Z/pZ$ and $M = S^n$ as G-spaces then the degree must be congruent one modulo p). We shall describe in this section the straighforward generalization to transformation groups, using the obstruction theory of 8.3.

We give the data needed to state the results. Let X be a G-CW-complex of finite orbit type. Then X^H is a WH-complex ($WH := NH/H$). We assume that all X^H are finite-dimensional. If H is an isotropy group of X we let n(H) be the dimension of X^H. For simplicity we assume that $n(H) \geq 1$. If $H \lneqq K$ then we should have $n(H) > n(K)$, for $H,K \in Iso(X)$ of course. We assume that $H^{n(H)}(X^H;Z) \cong Z$. The action of WH on X^H then induces a homomorphism $e_{H,X} : WH \longrightarrow Z^* = \{\pm 1\} = Aut\ Z$ which is called the orientation behaviour of X at H. We put $\bar{X}^H = \cup X^K$, $K \gneqq H$; this is a WH-subspace of X^H. The map $e_{H,X}$ defines a WH-module $Z_{H,X}$ which we use for local coefficients in order to define the group $H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X})$. We assume that this cohomology group is isomorphic to Z if WH is finite. But be have the

<u>Lemma 8.4.1. If under the assumption above</u> $n(H) \geq n(K)+2$ <u>for all</u> $K > H$, $K \neq H$, $K \in Iso(X)$ <u>then</u> $H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X}) \cong Z$.

<u>Proof.</u> Using the exact cohomology sequence of the pair we see that it

suffices to show that $H^{n(H)}(X^H/WH;Z_{H,X}) \cong Z$. We look at cellular cochains $\mathrm{Hom}_{WH}(C_{n(H)}(X^H), Z_{H,X})$. If $n(H) \geq n(K) + 1$ for $K > H$, $K \neq H$, $K \in \mathrm{Iso}(X)$ then $C_{n(H)}(X^H)$ is a free WH-module (for WH finite) hence the trace map which makes cochains WH-equivariant is surjective, hence the transfer $H^{n(H)}(X^H;Z) \longrightarrow H^{n(H)}(X^H/WH;Z_{H,X})$ is surjective. The composition of this map with the map in the other direction induced by $X^H \longrightarrow X^H/WH$ is multiplication by $|WH|$. So we only show that the group in question is torsion free. But one shows easily, using the trace operator that

$$\mathrm{Hom}_W(Z_H, Z_H) \longleftarrow \mathrm{Hom}_W(C_n, Z_H) \longleftarrow \mathrm{Hom}_W(C_{n-1}, Z_H)$$

is exact.

We now continue to describe data. Let Y be another G-space. We assume that Y^H is n(H)-connected and $\pi_{n(H)} Y^H \cong Z$ for $H \in \mathrm{Iso}(X)$. Then $H^{n(H)}(Y^H;Z) \cong Z$ and we obtain the orientation behaviour $e_{H,Y}: WH \longrightarrow Z^*$ of Y at H. We assume that $e_{H,X} = e_{H,Y}$ for all $H \in \mathrm{Iso}(X)$. We orient X be choosing a generator of $H^{n(H)}(X^H;Z)$ for every H and similarly for Y. We assume that X and Y have been oriented. Then given a G-map $f: X \longrightarrow Y$ the fixed point mapping $f^H : X^H \longrightarrow Y^H$ has a well-defined degree $d(f^H) \in Z$.

Theorem 8.4.1. *Under the assumption above the equivariant homotopy set $[X,Y]_G$ is not empty. Elements $[f] \in [X,Y]_G$ are determined by the set of degrees $d(f^H)$, $H \in$ Iso X, WH finite. The degree $d(f^H)$ is modulo $|WH|$ determined by the $d(f^K)$, $K > H$, $K \neq H$ and fixing these $d(f^K)$ the possible $d(f^H)$ fill the whole residue class mod $|WH|$.*

Proof. We order the isotropy types $(H_1),\dots,(H_r)$ of X such that $(H_i) < (H_j)$ implies $i > j$. Let $(H) = (H_i)$ and suppose that we already have a G-map $f : \bigcup_{j<i} GX^{H_j} =: X_{i-1} \longrightarrow Y$. We want to extend this G-map to X_i. As we have explained in 8.1 the homotopy classes rel X_{i-1} of such extension correspond to WH-extensions of $f|\bar{X}^H$ to X^H. The obstructions to such extensions lie in $H^i(X^H/WH, \bar{X}^H/WH; \pi_{i-1}(Y^H))$ and these groups are all zero by our assumptions. Hence there exists at least one extension.

Given two WH-maps $f,g : X^H \longrightarrow Y^H$ with $f|\bar{X}^H = g|\bar{X}^H$ the obstructions against a homotopy between them lie in the groups $H^i(X^H/WH, \bar{X}^H/WH; \pi_i(Y^H))$ and these groups are all zero except for $n(H) = i$ and WH finite where $\pi_{n(H)}(Y^H) = Z_{H,Y} = Z_{H,X}$ and the group is Z by assumption. Hence we get a single integer $d(f,g)$ as an obstruction. We claim that $d(f,g)$ is divisible by $|WH|$ and moreover $d(f,g) = d(f) - d(g)$. We look at the natural map

$$p^* : H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X}) \longrightarrow H^{n(H)}(X^H, \bar{X}^H; Z).$$

By naturality of the obstruction class $d(f,g)$ is mapped onto the obstruction against a non-equivariant homotopy between f and g and this is by the classical Hopf theorem just the difference of the degrees. We have already seen above that image $p^* \subset |WH|\, Z$. Together with 8.3.1, applied to this induction step, this finishes the proof of 8.4.1.

8.5. Geometric modules over the Burnside ring.

We shall prove in this section that the Burnside ring A(G) is isomorphic to stable cohomotopy of spheres in dimension zero via the Lefschetz-Dold index, see 7.6.7. The proof will be computational but gives at the same time information about certain other modules over A(G). We recall

Theorem 8.5.1. If we assign to a compact G-ENR X *the Lefschetz-Dold index* I(X) *we obtain a well-defined map* $I_G : A(G) \longrightarrow \omega_G^0$. *This map is an isomorphism of rings.*

Proof. If H is a closed subgroup of G we define a ring homomorphism $d_H : \omega_G^0 \longrightarrow Z$ by assigning to $x \in \omega_G^0$, represented by $f : V^c \longrightarrow V^c$, the degree of the H-fixed point map f^H. Recall that we introduced in section 5 a homomorphism $\varphi_H : A(G) \longrightarrow Z : [X] \longmapsto \chi(X^H)$, where χ denotes the Euler characteristic.

We show: Let X be a compact G-ENR. Then $d_H I(X) = \chi(X^H)$. By 7.6.8 we have $d_H I(X) = I(X^H) \in \omega_{\{1\}}^0 \cong Z$. The fixed point index $I(X^H)$ of $id(X^H)$ is the Euler characteristic of X^H (compare Dold [75] , XII 6.6 and [76]). This proves $d_H I(X) = \chi(X^H)$. By 8.4.1 the elements of ω_G^0 are detected by the maps d_H. From the definition of the Burnside ring we now obtain that I_G is a well-defined injective ring homomorphisms. That this map is also surjective will follow if we show that the $d_H(x)$ satisfy congruences analogous to 5.8.5. (See 8.5.9) We shall prove this in a moment for a slightly more general situation.

Remark 8.5.2. If $f : X \longrightarrow X$ is an endomorphism of the compact G-ENR X then the Lefschetz-Dold index of (X,f) is an element of $\omega_G^0 = A(G)$. By 5.5.1 this index element is a linear combination of homogeneous spaces. It is a non-trivial exercise for the reader to figure out which linear combination this is.

The isomorphism of Theorem 8.5.1 is natural, i. e. commutes with the various restriction and induction processes. If $f : G \longrightarrow K$ is a continuous homomorphism then we obtain by pull-back along f homomorphisms $f^* : \omega_K^0 \longrightarrow \omega_G^0$ and $f^* : A(K) \longrightarrow A(G)$ and we have

$$I_G\, f^* = f^* I_K.$$

The adjointness $[G^+ \wedge_H X, Y]^o_G \cong [X,Y]^o_H$ for a pointed H-space X and a pointed G-space Y together with the G-homeomorphism

$$G^+ \wedge_H X \longrightarrow G/H^+ \wedge X \; : \; (g,x) \longmapsto (g,gx)$$

for a G-space X induces an isomorphism

$$i^G_H \; : \; \omega^o_H \cong \omega^o_G(G/H).$$

If we compose this with the transfer induced by G/H $\longrightarrow$ Point we get the induction

$$\mathrm{ind}^G_H \; : \; \omega^o_H \longrightarrow \omega^o_G.$$

Note that we also have a map

$$I_{[G/H]} \; : A[G/H] \longrightarrow \omega^o_G(G/H)$$

which assigns to a submersion $f : M \longrightarrow G/H$ the Lefschetz-Dold index I_f. In 5.12 we constructed an isomorphism $i^G_H : A(H) \longrightarrow A[G/H]$.

<u>Proposition 8.5.3.</u> $I_{[G/H]}\, i^G_H = i^G_H\, I_H$

$$\mathrm{ind}^G_H\, i^G_H = i^G_H\, \mathrm{ind}^G_H.$$

<u>Proof.</u> This follows from properties 7.6.8 of the transfer.

Finally we mention that the maps I_H are compatible with the multiplicative induction. If H has finite index in G we showed in 5.12

that the multiplicative induction $X \longmapsto \mathrm{Hom}_H(G,X)$ induced a map $A(H) \longrightarrow A(G)$. This map is transformed under the isomorphisms I_H, I_G into a map $\omega_H^o \longrightarrow \omega_G^o$ which has the following description on representatives: Note that $\mathrm{Hom}_H(G,X)$ as a space is just $\prod(gH \times_H X)$, the product taken over the cosets G/H; but this formulation also indicates the G-action. If now X is a pointed H-space then we can similarly form the smashed product $\bigwedge(gH \times_H X)$ with G-action defined similarly. This gives a functor from pointed H-spaces to pointed G-spaces which maps H-homotopies to G-homotopies. If V is an H-module then $\bigwedge(gH \times_H V^c)$ is the one-point-compactification of the induced representation $\mathrm{Hom}_H(G,V)$. The map in question is now induced by $[V^c, V^c]_H^o \longrightarrow [\bigwedge(gH \times_H V^c), \bigwedge(gH \times_H V^c)]_G^o \longrightarrow [(\mathrm{Hom}_H(G,V))^c, (\mathrm{Hom}_H(G,V))^c]_G^o$. More generally, multiplicative induction is a map $\omega_H^o(X) \longrightarrow \omega_G^o(\mathrm{Hom}_H(G,X))$. The reader may check that multiplicative induction is compatible with the Lefschetz index.

Suppose now that we given complex representations V and W such that

(8.5.4) $$\dim V^H = \dim W^H \quad \text{for all } H < G.$$

We call $\omega_\alpha = \omega_o^G(V^c, W^c)$ the ω_G^o-module for $\alpha = V-W$. For each $H < G$ we have a degree map

(8.5.5) $$d_{\alpha,H} : \omega_\alpha \longrightarrow Z : [f] \longmapsto \text{degree } f^H.$$

The degree is computed with respect to the canonical orientations of $(V^H)^c$, $(W^H)^c$ which are induced by the complex structure. By 8.4.1 the maps $d_{\alpha,H}$ detect the elements of ω_α. So we ask: What are the relations between the possible degrees $d_{\alpha,H}(x)$? The assignment $(H) \longmapsto d_{\alpha,H}(x)$ is a continuous function. Therefore we obtain an injective map

(8.5.6) $$d_\alpha : \omega_\alpha \longrightarrow C(\phi, Z).$$

We want to describe the image by congruence relations.

Theorem 8.5.7. There exists a collection of integers $n_{H,K}(\alpha)$, depending on α, $(H) \in \phi(G)$, and (K) with H normal in K and K/H cyclic, such that $n_\alpha(H,H) = 1$ and such that the following holds: $x \in C(\phi, Z)$ is contained in the image of d_α if and only if:

$$\Sigma_{(K)} n_{H,K}(\alpha)\ x(K) \equiv 0 \mod |NH/H|.$$

The sum is taken over the conjugacy classes (K) such that H is normal in K and K/H is cyclic.

Proof. We first show that any set of congruence relations of the type considered in 8.5.7 suffices to the describe the module ω_α. Later we derive specific congruences as indicated, using K-theory.

Suppose we are given for each $(H) \in \phi$ a map $r_H : C(\phi, Z) \longrightarrow Z/|WH|$ of the form

(8.5.8) $$r_H(z) = z(H) + \Sigma\, n_{H,K}\ z(K) \mod |WH|$$

where the $n_{H,K}$ are integers and the sum is taken over the conjugacy classes (K) such that H is normal in K and K/H is a non-trivial cyclic group. Suppose that for $\alpha = E - F$ with $\dim E^H = \dim F^H$ the image of d_α is contained in

$$C_\alpha = \{ z \in C(\phi, Z) \mid (H) \in \phi \Rightarrow r_H(z) = 0 \}$$

Then we claim $d_\alpha \omega_\alpha = C_\alpha$.

Given $z \in C_\alpha$. We have to show that for a suitable U there exists a map $f : S(E \oplus U) \longrightarrow S(F \oplus U)$ such that for each $(H) \in \phi$ degree $f^H = z(H)$. To begin with we choose U large enough so as to satisfy the following conditions:

i) $Iso(E \oplus U) = Iso(F \oplus U)$

ii) $(1), (G) \in Iso(E \oplus U)$

iii) $(K), (L) \in Iso(E \oplus U) \Rightarrow (K \cap L) \in Iso(E \oplus U)$

iv) Choose an integer $n \neq 0$ such that $x = nz$ is contained in C_o. Then there shall exist a representative $S(E \oplus U) \longrightarrow S(F \oplus U)$ for $x \in \omega_o$.

Once (iv) is satisfied for U it is also satisfied for any U' containing U as a direct summand. Hence by enlarging U we can also satisfy (i) - (iii).

We set $X = S(E \oplus U)$ and $Y = S(F \oplus U)$. Let $Iso(X) = \{(H_1),\dots,(H_r)\}$ where $(H_i) > (H_j)$ implies $i < j$. If $X_i = \{x \in X \mid (G_x) = (H_j)$ for some $j \leq i\}$ we construct inductively G-maps $f_r : X_r \longrightarrow Y$ such that

v) degree $f_r^L = z(L)$ if $(L) \in \phi$, $(L) \geq (H_i)$, $i \leq r$
or if $(L) > (H_{r+1})$, $(L) \in \phi$.

Note that $X_r^L = X^L$ for such L. Put $H = H_{r+1}$. The G-extensions $f_{r+1} : X_{r+1} \longrightarrow Y$ of f_r correspond via restriction bijectively to the WH-extensions $h : X^H \longrightarrow Y^H$ of $f_r' = f_r \mid X_H : X_H \to Y^H$ where $X_H = X^H \cap X_r$. The obstructions to the existence of h lie in $H^*(X^H/HN, X_H/NH; \pi_{*-1}(Y^H))$, as in 8.4. These groups are zero by our assumptions. Let f_{r+1}' be a WH-extension of f_r'. Let $f_1 : X \longrightarrow Y$ be a map with $f_1^H = f_{r+1}'$ which exists by the same obstruction argument. Then, if $(H) \in \phi$, we have for the fixed point degrees

$$d(f_1^H) + \sum n_{H,K}\, d(f_1^K) \equiv 0 \bmod |WH| .$$

By induction $d(f_1^K) = z(K)$ so that in this case $d(f'_{r+1}) \equiv z(H) \bmod |WH|$. Since WH acts freely on $X^H \setminus X_H$ we can alter f'_{r+1} rel X_H to an NH-map f''_{r+1} so that $d(f''_{r+1}) = z(H)$. Let f_{r+1} be the map with $f_{r+1}| X^H = f''_{r+1}$ if $(H) \in \phi$ and $f_{r+1} | X^H = f'_{r+1}$ if $(H) \notin \phi$. Then $d(f^L_{r+1}) = z(L)$ whenever $(L) \geq (H_i)$, $i \leq r+1$. Suppose $(L) > (H_{r+1})$, $(L) \in \phi$. Since $Iso(X) = Iso(Y)$ is closed under intersections there exists a unique isotropy group $(P) = (H_s)$ such that $(P) \geq (L)$ and $(P) \in \phi$, $X^L = X^P$, $Y^L = Y^P$, degree f^L_{r+1} = degree $f^P_{r+1} = z(P)$. We have to show $z(L) = z(P)$. But by (iv) above nz is represented by a map $g : X \longrightarrow Y$ hence $g^P = g^L$ implies $nz(L) = nz(P)$. This finishes the proof of $d_\alpha \omega_\alpha = C_\alpha$.

We now derive specific functions of the type 8.5.8. Let $f : E \longrightarrow F$ be a proper G-map between complex G-modules. Let $C < G$ be a topologically cyclic group with generator h. Put $E = E^C \oplus E_C$, $j_E : E_C \subset E$. We apply equivariant K-theory with compact support and obtain for $f^*: K_C(F) \longrightarrow K_C(E)$ and $(f^C)^*$ the equality $j_E^* f^* = j_F^* (f^C)^*$. Let $\lambda(E) \in K_G(E)$ be the Bott class, a free R(G)-generator of $K_G(E)$. Then we define $a \in R(G)$ by $f^* \lambda(E) = a \lambda(F)$ and obtain $(a|C) \lambda_{-1}(E_C) = \lambda_{-1}(F_C)$ degree f^C. We evaluate characters at h and use $\lambda_{-1}(E_C)(h) \neq 0$. If G is finite then $\sum_{g \in G} a(g) \equiv 0 \bmod |G|$. If $C < G$ is cyclic and C^* its set of generators we put $a^*(C) = \sum_{g \in C^*} a(g)$. With $n(E-F,C) = \sum_{g \in C^*} \lambda_{-1}(F_C)(g)/\lambda_{-1}(E_C)(g)$ we obtain

$$a^*(C) = n(E-F,C) \text{ degree } f^C$$

$$a(g) = \sum_{(C)} |G|\, |NC|^{-1}\, a^*(C) \equiv 0 \bmod |G|.$$

By elementary Galois theory $n(E-F,C)$ is an integer. We apply these considerations to f^H considered as WH-map and obtain

$$\Sigma_K \; |NH/NH \cap NK| \; n(E^H - F^H, K/H) \; d(f^K) \equiv o \bmod |WH|$$

where the sum is taken over the NH-conjugacy classes (K) with $H \triangleleft K$ and K/H cyclic. This yields the desired functions 8.5.8.

Remark 8.5.9. Comparing the case E = F of the above congruences with 5.8.5 we see that the map I_G of 8.5.1 is surjective.

8.6. Prime ideals of equivariant cohomotopy rings.

Let X be a compact G-ENR, G finite. We are going to determine the prime ideal spectrum of the ring $\omega_G^o(X)$.

The orbit category O(X) of X shall have as objects the G-homotopy classes of maps $G/H \longrightarrow X$ and as morphisms from $u : G/H \longrightarrow X$ to $v : G/K \longrightarrow X$ the G-homotopy classes $t : G/H \longrightarrow G/K$ such that $vt = u$.

If $u : G/H \longrightarrow X$ is given we have the induced ring homomorphism $u^* : \omega_G^o(X) \longrightarrow \omega_G^o(G/H)$ and the maps u^* combine to a ring homomorphism

$$(8.6.1) \qquad \nu : \omega_G^o(X) \longrightarrow \lim \omega_G^o(G/H)$$

where the limit (= inverse limit) is taken over the category O(X). Let Spec ν be the induced map of prime ideal spectra.

Theorem 8.6.2. The kernel of ν is the nilradical of $\omega_G^o(X)$. For each $x \in \lim \omega_G^o(G/H)$ there exists an $n \in \mathbb{N}$ with $x^n \in$ image ν. The map ν induces a homeomorphism Spec ν of prime ideal spectra.

Next we show that taking prime ideal spectra commutes with taking limits over the category $O(X)$. The canonical maps $\lim \omega_G^0(G/H) \to \omega_G^0(G/H)$ induce a continuous map μ: colim Spec $\omega_G^0(G/H) \longrightarrow$ Spec lim $\omega_G^0(G/H)$.

<u>Theorem 8.6.3. The map μ is a homeomorphism.</u>

We now enter the proofs of these Theorems.

Recall that one has Bredon cohomology [36] $H^*(X;\omega)$ of X with coefficient system ω given by $\omega : G/H \longrightarrow \omega_G^0(G/H)$ on objects and induced maps (see also Bröcker [38] or Illman for an exposition of this cohomology theory). Let

$$e : \omega_G^0(X) \longrightarrow H^0(X;\omega)$$

be the edge-homomorphism associated to the Atiyah-Hirzebruch spectral sequence of $\omega_G^0(-)$. More directly: $H^0(X;\omega)$ is canonically isomorphic to $\lim \omega_G^0(G/H)$ and under this isomorphism e corresponds to ν.

<u>Proposition 8.6.4.</u> (i) <u>The map $e \otimes Q$ is an isomorphism.</u>
(ii) <u>The torsion subgroup of $\omega_G^0(X)$ as abelian group is equal to the nilradical of the ring $\omega_G^0(X)$.</u>

<u>Proof.</u> (i) If $e \otimes Q$ is an isomorphism for a space X then also for any G-retract of X. Since any G-ENR is a retract of a finite G-CW-complex (dominated by a finite G-CW-complex suffices and this is easier to see) it is enough to consider finite G-CW-complexes. But e is a natural transformation of half-exact homotopy functors, so by a standard comparison theorem (see e. g. Dold [72]) it suffices to show that $e \otimes Q$ is an isomorphism on cells. This is true for zero-cells by the very definition of $H^0(X;\omega)$. If $i > 0$ then $H^0(G/H \times (D^i, S^{i-1});\omega) = 0$ by the dimension axiom of this equivariant cohomology theory. On the other

hand

$$\omega_G^o(G/H \times (D^i, S^{i-1})) \cong \omega_H^o(D^i, S^{i-1}) \cong \omega_i^H$$

and by the splitting theorem of Segal [145] , (see also tom Dieck [63] , Satz 2) we have

$$\omega_i^H \cong \oplus_{(K)} \omega_i (BWK^+)$$

(the product is over conjugacy classes (K) of subgroups of H; WK=NK/K, NK normalizer of K in H). But $\omega_i(BWK^+)$ is for i > o a torsion group.

(ii) The kernel of e is the nilradical of $\omega_G^o(X)$. The nilradical is certainly contained in this kernel because $H^o(X;\omega)$ is contained in in product of rings of the type $\omega_G^o(G/H)$ and these rings have no (non-zero) nilpotent elements (being isomorphic to the Burnside ring A(H).) On the other hand the kernel consists precisely of elements of skeleton filtration one hence consists of nilpotent elements. (See Segal [142] for an analogous statement.) Since $H^o(X;\omega)$ is torsion-free we have Torsion $\omega_G^o(X) \subset$ Nil $\omega_G^o(X)$. Tensoring the exact sequence

$$0 \longrightarrow \text{Nil}\ \omega_G^o(X) \longrightarrow \omega_G^o(X) \longrightarrow H^o(X;\omega)$$

with Q and using (i) we obtain (ii).

Note that Proposition 8.6.4 proves the first statement of Theorem 8.6.2. We now come to the second statement.

<u>Proposition 8.6.5.</u> *The map e : $\omega_G^o(X) \longrightarrow H^o(X;\omega)$ has "nilpotent cokernel", i. e. a suitable power of every element of $H^o(X;\omega)$ is contained in the image of e.*

<u>Proof.</u> (Compare Quillen [127]). If the assertion of the Proposition is true for X then also for any G-retract of X. Since X is a compact G-ENR it is a retract of a compact differentiable G-manifold with boundary. So we need only prove the Proposition for those X which are locally contratible (i. e. each orbit of X is a G-deformation retract of a neighbourhood). If X is G-homotopy equivalent to an orbit then the map e is an isomorphism. Now assume that $X = U_1 \cup \ldots \cup U_n$, the U_i being compact G-ENR's which are G-homotopy equivalent to an orbit. Assume that the Proposition is true for $X_1 = U_1 \cup \ldots \cup U_{n-1}$. We consider the following diagram of Mayer-Vietoris sequences where $H^o(X) = H^o(X;\omega)$ and e_i are instances of the transformation e.

$$\begin{array}{ccccccc}
 & & \omega_G^o(X) & \xrightarrow[t]{} & \omega_G^o(X_1) \oplus \omega_G^o(U_n) & \xrightarrow[s]{} & \omega_G^o(X_1 \cap U_n) \\
 & & \Big\downarrow e & & \Big\downarrow e_1 \oplus e_2 & & \Big\downarrow e_3 \\
o & \longrightarrow & H^o(X) & \xrightarrow[t']{} & H^o(X_1) \oplus H^o(U_n) & \xrightarrow[s']{} & H^o(X_1 \cap U_n)
\end{array}$$

Given $x \in H^o(X)$ we put $t'(x) = (x_1, x_2)$. By induction hypothesis there exists k such that

$$t'x^k = (x_1^k, x_2^k) = (e_1u_1, e_2u_2)$$

for suitable u_i. By exactness $s'x_1^k = s'x_2^k$ hence $su_2 = su_1 + n$, where n is a suitable nilpotent element by Proposition 8.6.4. Suppose $n^1 = 0$. Then for $p > t$, with $z = su_1$,

$$(z+n)^p = z^p + \binom{p}{1} z^{p-1} n + \ldots + \binom{p}{t-1} z^{p-t+1} n^{t-1}.$$

By Proposition 8.6.4 the elements $n, n^2, \ldots, n^{t-1}$ are torsion elements. Choose $q \in \mathbb{N}$ such that $qn^i = 0$ for $1 \leq i \leq t-1$. Choose p such that q divides $\binom{p}{1},\ldots,\binom{p}{t-1}$, e. g. $p = (t-1)!q$. Then we obtain

$$(z+n)^p = z^p ,$$

i. e.

$$(su_1)^p = s(u_1^p) = s(u_2^p)$$

and we can find y with $ty = (u_1^p, u_2^p)$, so that finally $fy = x^{pk}$. This proves the induction step.

The final assertion of Theorem 8.6.2 comes from commutative algebra. We have the following situation: $A \xrightarrow[f]{} A/\mathrm{Nil}\,A \xrightarrow[g]{} B$ where f is the canonical quotient map and g is an injection with nilpotent cokernel. Then Spec f is a homeomorphism. Since g has nilpotent cokernel it is easy to see that Spec g is injective. On the other hand g is an integral extension; by the going up theorem Spec g is a closed surjective mapping. Hence also Spec g is a homeomorphism in our case. This finishes the proof of Theorem 2.

Theorem 8.6.4 is contained in Quillen [127], Corollary B.7 in the Appendix B.

We are going to give more explicit statements for some of the results above. Let $x \in X$ and let $H < G_x$ be a subgroup of the isotropy group at x. We define a ring homomorphism $\varphi_{x,H} : \omega_G^0(X) \longrightarrow Z$ as the composition

$$\omega_G^0(X) \longrightarrow \omega_H^0(X) \longrightarrow \omega_H^0(\{x\}) \cong A(H) \longrightarrow Z$$

where the first two maps are restrictions and the last one takes the degree or Euler characteristic of the H-fixed point object.

<u>Proposition 8.6.6. Every ring homomorphism</u> $\varphi : \omega_G^0(X) \longrightarrow Z$ <u>is of the</u>

<u>form</u> $\varphi_{x,H}$ <u>for suitable</u> $x \in X$ <u>and</u> $H < G_x$. <u>We have</u> $\varphi_{x,H} = \varphi_{y,K}$ <u>if and only if</u> $(H) = (K)$ <u>and</u> x <u>and</u> y <u>are in the same orbit under</u> WH <u>of the path-components of</u> X^H. <u>The prime ideals of</u> $\omega_G^0(X)$ <u>have the form</u> $\varphi_{x,H}^{-1}(p)$, $(p) \subset Z$ <u>a prime ideal</u>.

<u>Proof.</u> Let q be the kernel of φ. This is a prime ideal which by Theorem 8.6.2 and 8.6.3 is equal to the kernel of some $\varphi_{x,H}$. Therefore we must have $\varphi = \varphi_{x,H}$.

The different homomorphisms $\varphi : \omega_G^0(X) \longrightarrow Z$ correspond bijectively to the minimal prime ideals of $\omega_G^0(X)$ and bijectively to the homomorphism $\omega_G^0(X) \otimes Q \longrightarrow Q$ of Q-algebras. But by the results of section 7 we have a natural ring isomorphism

$$\omega_G^0(X) \otimes Q \cong \bigoplus_{(H)} \omega^0(X^H)^{WH} \otimes Q$$

where the sum is over the conjugacy classes (H) of subgroups $H < G$. From this fact one easily deduces the second statement of the Proposition. The third one is again a restatement of the Theorems above.

8.7. Comments.

This section is rather rudimentary. We give some references to further developments. A detaild discussion of the Hopf theorem 8.4.1 for maps between spheres can be found in Hauschild [93] . A more conceptual proof of 8.5.1 uses splitting theorem of tom Dieck [63] , Satz 2. Other splitting theorems may be found in Segal [145] , Rubinsztein [136] , Kosniowski [105] , Hauschild [90] , [93] ; relevant is also Wirthmüller [168] and Schultz [138] . 8.5.7 has been generalized to unstable and real modules by Tornehave [160] . 8.2 is based on Hauschild [94] and Vogt [23] , Appendix. For the use of obstruction theory as in 8.3 to equivariant versions of the Blakers-Massey theorem and the suspension theorem see Hauschild [92] . 8.6 was presented in lectures by the author in Newcastle-upon-Tyne, April 1975; also the double coset formula for the equivariant transfer (see exercises).

8.8. Exercises.

1. Show that the double coset formula of 5.12 holds in equivariant cohomotopy and hence in any stable equivariant cohomology of homology theory. (This genralizes various results in Feshbach [82] , Brumfiel-Madsen [43] etc.) More specifically: Let $x_M \in \omega^G_o(M)$ be the transfer element corresponding to $M \longrightarrow$ Point. Let $M = \Sigma\, n_{(H),b}\, M_{(H),b}$ with $n_{(H),b} = \chi_c(S_{(H),b}/G)$ be the decomposition in the Burnside ring as in 5.12. Let $x_{(H),b} \in \omega^G_o(M_{(H),b})$ be the transfer element corresponding to $M_{(H),b} \longrightarrow$ Point. Let $i_{(H),b} : \omega^G_o(M_{(H),b}) \longrightarrow \omega^G_o(M)$ be induced by the inclusion. Then show

$$x_M = \Sigma\, n_{(H),b}\, i_{(H),b}(x_{(H),b}).$$

2. Let $H < G$ and let L be the tangent space of G/NH at 1. Show that there exists a natural isomorphism

$$\omega^{NH}_n(L^c \wedge EW^+ \wedge X) \longrightarrow \omega^G_n((G \times_N EW)^+ \wedge X),$$

$n \in Z$.

3. (tom Dieck [63]) Show that there exists a natural isomorphism

$$\oplus_{(H)}\, \omega^{WH}_n\, (EWH^+ \wedge X^H) \longrightarrow \omega^G_n(X),$$

$n \in Z$, G compact Lie group, the sum over conjugacy classes of subgroups.

9. Homotopy Equivalent Group Representations.

We are concerned in this section with the homotopy theory of group representations. If G is a compact Lie group and E and F are orthogonal real representations so that the unit spheres S(E) and S(F) are preserved by the G-action, we ask: When does there exist a G-map $f : S(E) \longrightarrow S(F)$ which has a G-homotopy inverse?

It turns out that homotopy equivalences between different representations can essentially only occur for finite groups. Therefore we shall only consider finite groups and restrict our attention to stable homotopy equivalences. Later we shall deal with the unstable situation and compact Lie groups.

9.1. Notations and results.

Let G be a finite group. If V is a (real or complex) G-module we denote by S(V) its unit sphere with respect to some G-invariant inner product. Two real G-modules V and W are called homotopy equivalent if the G-spaces S(V) and S(W) are G-homotopy equivalent. If V and W (resp. V_1 and W_1) are homotopy equivalent, then $V \oplus V_1$ and $W \oplus W_1$ are homotopy equivalent because $S(V \oplus V_1)$ is G-homeomorphic to the join $S(V) * S(V_1)$ and we can use the join of the individual homotopy equivalences. Two real G-modules V and W are called stably homotopy equivalent if for some real G-module U the modules $V \oplus U$ and $W \oplus U$ are homotopy equivalent. Let R(G) resp. RO(G) denote the Grothendieck ring of complex resp. real G-modules (identified with the corresponding character ring). Elements $x \in RO(G)$ are formal differences $x = V-W$ of real G-modules V and W. The $x = V-W$ such that V and W are stably homotopy equivalent form, by the remark above about joins, an additive subgroup of RO(G), denoted $RO_h(G)$. If we deal with complex G-modules we call V and W oriented homotopy equivalent if there exists a G-homotopy equivalence

$f : S(V) \longrightarrow S(W)$ such that for each subgroup H of G the induced map $f^H : S(V)^H \longrightarrow S(W)^H$ on the H-fixed point sets has degree one with respect to the coherent orientations that $S(V)^H$ and $S(W)^H$ inherit from the complex structure on V^H and W^H. We let $R_h(G)$ be the additive subgroup of R(G) consisting of $x = V-W$ such that V and W are oriented stably homotopy equivalent.

If $S(V \oplus U)$ and $S(W \oplus U)$ are G-homotopy equivalent then the H-fixed points are homotopy equivalent. In particular the spheres $S(V)^H$ and $S(W)^H$ then have the same dimension (or are both empty). Let $R_o(G)$ be the additive subgroup of the V-W such that for all subgroups $H < G$ we have $\dim V^H = \dim W^H$. Let $RO_o(G)$ be the analogous subgroup of RO(G). Since $R_h \subset R_o$ and $RO_h \subset RO_o$ we introduce the groups

(9.1.1) $$j(G) = R_o(G)/R_h(G), \quad jO(G) = RO_o(G)/RO_h(G).$$

If G has order $g = |G|$ then G-modules are realisable over the field Q(u) where u is a primitive g-th root of unity. The Galois group Γ of Q(u) over Q acts on R(G) and RO(G) via its action on character value (see 3.5). Actually Γ acts on the set

$$\mathrm{Irr}\ (G,C) \quad \text{resp.} \quad \mathrm{Irr}\ (G,\mathbb{R})$$

of complex resp. real irreducible G-modules. Let $Z[\Gamma]$ be the integral group ring of Γ and $I(\Gamma)$ its augmentation ideal. Then we have

<u>Proposition 9.1.2. The following equalities hold</u>

$$R_o(G) = I(\Gamma)R(G), \quad RO_o(G) = I(\Gamma)RO(G).$$

The need for the following objects will become clear in a moment:

$$(9.1.3) \qquad R_1(G) = I(\Gamma)R_o(G), \quad RO_1(G) = I(\Gamma)RO_o(G)$$
$$i(G) = R_o(G)/R_1(G), \quad iO(G) = RO_o(G)/RO_1(G) .$$

We shall obtain the following results.

Theorem 9.1.4. For all finite groups G *we have*

$$R_1(G) \subset R_h(G) \quad \text{and} \quad RO_1(G) \subset RO_h(G).$$

Using this theorem we can consider the canonical quotient maps

$$(9.1.5) \qquad t(G) : i(G) \qquad j(G), \quad tO(G) : iO(G) \qquad jO(G).$$

Theorem 9.1.5. Let G *be a* p-*group. Then* t(G) *and* tO(G) *are isomorphisms.*

The plan of the demonstration of 9.1.4 and 9.1.6 is as follows: We begin with a recollection of some representation theory in 9.2, proving 9.1.2 and giving a detailed analysis of i(G) and iO(G). In 9.3 we shall prove 9.1.4 and in 9.6 we shall prove 9.1.6 using the functorial properties of 9.1.5. In subsequent section we discuss various extensions and refinements: Nilpotent and hyperelementary groups; maps between unstable modules; connections with the Burnside ring and rational characters.

9.2. Dimension of fixed point sets.

The number of irreducible complex representations of G equals the number of conjugacy classes of elements of G (see Serre [147], Théorème 7), in symbols

$$|\mathrm{Irr}\ (G,C)| = |\mathrm{Conj}\ (G)| .$$

Let $\Gamma = \Gamma(m)$ be the Galois group of Q(u) over Q where u is a primitive m-th root of unity and m is a multiple of $|G|$. The group Γ may be identified with the group of units in the ring Z/m. The group Γ acts on Irr(G,C). Let $X = X(G) = \mathrm{Irr}(G,C)/\Gamma$ be the orbit set of this action (it is independent of m). Then the elements

$$x_A = \Sigma_{y \in A}\, y, \quad A \in X(G)$$

form a Z-basis of the invariants

(9.2.1) $$R(G)^{\Gamma}.$$

The rational representation ring R(G;Q) is contained in $R(G)^{\Gamma}$ as a subgroup of maximal rank but in general different from it. There exists an integer n_A (the Schur-index, see 9. 3.) such that $n_A x_A$ is represented by an irreducible rational representation (Serre [147], 12.) Hence

(9.2.2) $$|X(G)| = \mathrm{Rank}_Z\, R(G;Q)$$

and this rank is equal to the number of conjugacy classes of cyclic subgroups (Serre [147], Théorème 29). Let $\zeta(G)$ be the set of conjugacy classes of cyclic subgroups of G and let $C(\zeta(G),Z)$ be the ring of functions $\zeta(G) \longrightarrow Z$. We obtain an additive map

(9.2.3) $$d : R(G) \longrightarrow C(\zeta(G),Z)$$
$$d(x)(C) = \dim_C x^C.$$

Since $\dim V^H = |H|^{-1} \Sigma_{h \in H} V(h)$ and the left hand side is Galois invariant we see that $I(\Gamma)R(G) \subset R_O(G) \subset$ kernel d. Hence we obtain a surjection

(9.2.4) $$R(G)_{\Gamma} := R(G)/I(\Gamma)R(G) \longrightarrow R(G)/\mathrm{Ker}\, d$$

which is compatible with the restriction to subgroups.

Proposition 9.2.5. The map 9.2.4 is injective, i. e.

$$I(\Gamma)R(G) = R_o(G) = \{V-W \mid \dim V^C = \dim W^C,\ C < G \text{ cyclic}\} .$$

Proof. We show that

$$R(G)_\Gamma \longrightarrow \prod_C R(C)_\Gamma$$

is injective, where C runs through the cyclic subgroups of C and the map is restriction. The group $R(G)_\Gamma$ is free abelian, a basis consisting of representatives for the Γ-orbits $Irr(G,C)/\Gamma$. The assignement $x \longmapsto \sum_{\gamma \in \Gamma} \gamma x$ induces a homomorphism $t : R(G)_\Gamma \longrightarrow R(G)$ which, composed with $R(G) \longrightarrow R(G)_\Gamma$, is multiplication by $|\Gamma|$. Hence t is injective. Since $R(G) \longrightarrow \prod R(C)$ is injective the map above must be injective. We now have a commutative diagram

$$\begin{array}{ccc} R(G) & \longrightarrow & \prod R(C)_\Gamma \\ \downarrow & & \downarrow \\ R(G)/\mathrm{Ker}\, d & \longrightarrow & \prod R(C)/\mathrm{Ker}\, d \end{array}$$

and it remains to be shown that for cyclic C the map $R(C) \longrightarrow R(C)/\mathrm{Ker}\, d$ is injective which is easily done by the reader.

Exactly the same argument shows

Proposition 9.2.6. For every finite group G

$$I(\Gamma)RO(G) = RO_o(G) = \{V-W \mid \dim V^C = \dim W^C,\ C < G \text{ cyclic}\} .$$

We therefore obtain from 9.2.3 and its real analog injective maps

(9.2.7)
$$d : R(G)_{\Gamma} \longrightarrow C(\zeta(G), Z)$$
$$dO : RO(G)_{\Gamma} \longrightarrow C(\zeta(G), Z)$$

with image group of maximal rank, i. e. the cokernel is a finite group. We want to compute the order of the cokernel. It would be interesting to know the actual structure of the cokernel.

We begin with a series of reductions. Let $V_1, \ldots, V_r$ be a system of representatives of $Irr(G,C)/\Gamma$ and $H_1, \ldots, H_r$ a system of representatives for $\zeta(G)$. Then

(9.2.8)
$$|Cok\ d| = \det(a_{ij})$$
$$a_{ij} = \dim Fix\ (H_j, V_i).$$

Using $|H| \dim V^H = \Sigma_{h \in H} V(h)$ we obtain

(9.2.9)
$$|Cok\ d| \prod_j |H_j| = |\det(\Sigma_{h \in H_j} V_i(h))|.$$

Let H^* denote the set of generators of the cyclic group H.

Lemma 9.2.10. *We have*

$$\det(\Sigma_{h \in H_j} V_i(h)) = \det(\Sigma_{h \in H_j^*} V_i(h)) .$$

Proof. Choose an indexing such that $(H_i) \leq (H_k)$ implies $k \leq i$. Put $b^*_{ij} = \Sigma_{h \in H_j^*} V_i(h)$ and $b_{ij} = \Sigma_{h \in H_j} V_i(h)$. Then

$$b_{ij} = b^*_{ij} + \Sigma_{l < j}\ e_l\ b_{il}$$

where $e_1 = 1$ or 0, independent of i. Subtracting suitable "earlier" columns from "later" one's we can transform the matrix (b_{ij}) into (b^*_{ij}).

We now observe that we can identify $\Gamma = Z/m^*$ in such a way that

$$\gamma V(g) = V(g^{\gamma})$$

so that Γ acts on each set H^*_j. We choose for each j an element $g_j \in H^*_j$ and let Γ_j be the isotropy group of the Γ-action at g_j. Then

$$(9.2.11) \qquad b^*_{ij} = |\Gamma_j|^{-1} \sum_{\gamma \in \Gamma} \gamma V_i(g_j).$$

Hence, if we put $IV = \sum_{\gamma \in \Gamma} \gamma V$, then we obtain from 9.2.10 and 9.2.11

$$(9.2.12) \qquad \det(b_{ij}) \prod_j |\Gamma_j| = \det(IV_i(g_j)).$$

In order to compute this determinant we make the following remark: Let W be a complex vector space with hermitian form $\langle -,- \rangle$ and orthogonal basis $e_1,\ldots,e_r$. Given $a_i = \sum c_{ik} e_k$, $1 \leq i \leq r$, then

$$(9.2.13) \qquad \det \langle a_i, a_j \rangle = (\det(c_{ik}))^2 \prod_j \langle e_j, e_j \rangle .$$

We shall compute $\det^2(IV(g_j))$ in this way. Consider IV_i as function on G. Put

$$G = C_1 \cup \ldots \cup C_r$$

where $g \in C_j$ if and only if g generates a group conjugate to H_j. Then IV_j belongs to the space of functions which are constant along the sets

C_j. Denote the characteristic function of C_j with the same letter. Then

(9.2.14) $$IV_i = \Sigma_j \, IV_i(g_j) \, C_j \, .$$

We use the standard hermitian form on the space of functions $G \longrightarrow \mathbb{C}$. Then $\langle C_j, C_j \rangle = |C_j|$. Using 9.2.13 we get

(9.2.15) $$(\Pi_j \, C_j) \det^2 (IV_i(g_j)) = \det \langle IV_i, IV_j \rangle \, .$$

The orthogonality relations for characters yield

(9.2.16) $$\langle IV_i, IV_j \rangle = G \, |\Gamma| \, |\Gamma^i| \, \delta_{ij}$$

where Γ^i is the isotropy group of the Γ-action on Irr(G,C) at V_i. Collecting our results we obtain

$$|\mathrm{Cok}\ d| = \Pi_j |H_j|^{-1} \cdot |\det(b_{ij})| \qquad (9.2.9)$$

$$= \Pi_j (|H_j|\,|\Gamma_j|)^{-1} \cdot |\det IV_i(g_j)| \qquad (9.2.12)$$

$$= \Pi_j (|H_j|\,|\Gamma_j|\,|C_j|^{1/2})^{-1} \, |\det \langle IV_i, IV_j \rangle| \qquad (9.2.15)$$

$$= \frac{|\Gamma|^{r/2} |G|^{r/2} \, \Pi |\Gamma^j|^{1/2}}{\Pi (|H_j| \, |\Gamma_j| \, |C_j|^{1/2})} \qquad (9.2.16)$$

If we note that $|\Gamma_j| \, |H_j^*| = |\Gamma|$ and $|C_j| = |H_j^*| \, |G/NH_j|$ we finally obtain

<u>Proposition 9.2.17.</u>

$$|\mathrm{Cok}\ d| = \frac{\Pi \, |NH_j|^{1/2}}{\Pi \, |H_j|} \cdot \frac{\Pi \, |\Gamma^j|^{1/2}}{\Pi \, |\Gamma_j|^{1/2}}$$

It is not obvious a priori that the right hand side of 9.2.17 is an integer. In certain cases the formula simplifies. The Γ-factors disappear for abelian groups G.

Proposition 9.2.18. Let G be a p-group, $p \neq 2$. Then Irr(G,C) and Conj(G) are isomorphic Γ-sets.

Proof. Let V_1 and V_2 be the permutation representations associated to the Γ-sets Irr(G,C) and Conj(G), respectively. We show that V_1 and V_2 are isomorphic Γ-representations. Since in our case Γ is cyclic and for such groups $A(\Gamma) \longrightarrow R(\Gamma)$ is injective we conclude that the Γ-sets in question are isomorphic. The isomorphism of V_1 and V_2 is given by identifying linear combination of elements of Irr(G,C) as usual with functions Conj(G) $\longrightarrow$ C. The formula 3.5.1 for the action of the Adams operations on characters shows that this is an isomorphism of Γ-modules.

If Γ_j' denotes the isotropy group of the conjugacy class of g_j and ZH_j the centralizer of H_j in G then

(9.2.19) $$|\Gamma_j'| \; |ZH_j| = |NH_j| \; |\Gamma_j| .$$

Using 9.2.17 - 19 we obtain

Proposition 9.2.20. Let G be a p-group, $p \neq 2$ a prime. Then the order of the cokernel of d is

$$\prod_j |NH_j/H_j| \; |ZH_j|^{-1/2} .$$

Let $c : R(G;Q) \longrightarrow C(\zeta(G),Z)$ be the ring homomorphism which associates with each $Q[G]$-module V the function c(V) such that c(V)(C) is the value of the character V at a generator of C. This is an inclusion of maximal rank. One would like to compute the cokernel; this would give congruences expressing conditions for functions to be a rational characters. Arguments as in the proof of 9.2.17 allow to compute the order of cokernel c. Let n_i be the Schur index of V_i.

Propositione 9.2.21. $|\mathrm{Cok}\ c| = \prod_j n_j\, |NH_j|^{1/2}$.

Proof. $|\mathrm{Cok}\ c| = |\det W_i(g_j)|$ where $W_j = n_j\ |\Gamma^j|^{-1}\ IV_j$ is the irreducible rational representation belonging to V_j. Now use the calculations above.

Problem 9.2.22. Compute the groups Cok c and Cok d. (The results of section 10 should be helpful.)

9.3. The Schur index.

We collect the classical results about the Schur index with emphasis on p-groups. We always work with subfields of the complex numbers. General references for the following are: Lang [107], Ch XVII; Curtis-Reiner [48], § 70; Roquette [135].

Let $k \subset \mathbb{C}$ be a field. The group algebra $k[G]$ is semi-simple and decomposes into a product of simple algebras A_i

$$k\,[G] = A_1 \oplus \dots \oplus A_r .$$

The corresponding decomposition $1 = e_1 + \dots + e_r$ yields the indecomposible central idempotents e_i of $k[G]$. By the theorem of Wedderburn each A_i is isomorphic to a full matrix algebra

$$A_i = M_{n_i}(D_i)$$

over a division algebra D_i. If V_i is a minimal left ideal of A_i, then V_i is an irreducible $k[G]$-module and every irreducible $k[G]$-module is isomorphic to one of this form. The endomorphismring of V_i is a division algebra, and in fact

$$D_i = \mathrm{Hom}_{k[G]}(V_i, V_i) .$$

The degree of D_i over its center K_i is a square m_i^2 where $m_i = [E_i, K_i]$ and E_i is a maximal field contained in K_i. The integer m_i is called the <u>Schur</u> <u>index</u> of V_i or A_i.

If V is an irreducible k(G)-module we let

$$A_V = A = \text{image } (k(G) \longrightarrow \mathrm{Hom}_k (V,V))$$

be the k-algebra generated by maps $l_g : v \longmapsto gv$. Then V is a faithfull irreducible A-module and since A is semisimple (being a quotient of $k[G]$) A must be simple. Hence $A = M_n(D)$ for some division algebra D whose center contains k.

If A is a simple algebra with center k then an extension field E of k is called a <u>splitting</u> <u>field</u> <u>for</u> A if $A \otimes_k E$ is a full matrix algebra over E. If A is a matrix algebra over the division algebra D then E is a splitting field if and only if E is a splitting field for A. If $[D:k]$ is finite then a maximal subfield E of D is a splitting field for D and $[D : k] = [E : k]^2$. If L is any other splitting field for D which is a finite algebraic extension of k then $[E : k]$ divides $[L : k]$.

Applying these results to the algebra $A = A_V$ above, assuming that k is the center of A (= center of D), then for any splitting field F of D one has

$$A \otimes_k F \cong M_{mn}(F)$$

where $m^2 = [D : k]$, $n^2 = [A : D]$. If U is an irreducible F(G)-module given by a minimal left ideal $A \otimes_k F$ then

$$V \otimes_k F \cong m\,U$$

which shows that mU is realisable over k. If tU is realisable over k then $m|t$.

If U is an irreducible $\mathbb{C}[G]$ -module we let $A_{k,U}$ be the k-algebra spanned by the $1_g \in \mathrm{Hom}_{\mathbb{C}}(U,U)$ which is a simple k-algebra. The center of this algebra is $k(\chi_U)$, this meaning k with character values $\chi_U(g)$ adjoined. The representation U is realisable over $F \supset k(\chi_U)$ if and only if F is a splitting field for $A_{k,U}$. The Schur index of $A_{k,U}$ is the minimal value m such that mU is realisable over $k(\chi_U)$ and there exists an extension F of degree m of $k(\chi_U)$ such that U is realisable over F. We therefore call $m = m_k(U)$ the _Schur index of U with respect to_ k.

We call E a _splitting field for_ G if every irreducible $\mathbb{C}[G]$ -module is realisable over E. If k is given one can always find a finite algebraic extension E of k which is a splitting field for G. By a famous theorem of Brauer $E = Q(u)$ is a splitting field for G if u is a primitive m-th root of unity and m is the last common multiple of the orders of elements in G.

Let V be an irreducible $k[G]$ -module. Let E be a splitting field for G which is a finite Galois extension of k. Then $V \otimes_k E$ splits

$$V \otimes_k E = m(U_1 \oplus \dots \oplus U_t)$$

where the U_i are irreducible $E[G]$-modules. Moreover $U_i \otimes_E \mathbb{C}$ is an irreducible $\mathbb{C}[G]$-module and $m = m_k(U_i \otimes_E C)$ for $i = 1,\dots,t$. The $U_1,\dots,U_t$ form an orbit under the action of the Galois group $\mathrm{Gal}(E : k)$ on the irreducible $E[G]$-modules. The number t above equals $k(\chi_1):k$ where χ_1 is the character of U_1.

For later reference we now collect what happens for p-groups. We follow Roquette [135].

Proposition 9.3.1. *Let G be a p-group. Then for each irreducible $\mathbb{C}[G]$-module V:*

i) *If* $p \neq 2$ *then* $m_Q(V) = 1$.

ii) *If* $p = 2$ *then* $m_Q(V) = m_{\mathbb{R}}(V)$ *is* 1 *or* 2.

Proof. Roquette [135] shows i) and $m_Q(V) = 1$ or 2. We make the additional remark that $m_Q = m_{\mathbb{R}}$. (This was communicated by J. Tornehave.) Roquette shows that in the case $m_Q(V) = 2$ the division algebra associated to $A_{Q,V}$ (in the notation above) is the ordinary quaternionic extension of its center $Q(\chi_V)$. Since $A_{Q,V} \otimes_{Q(\chi_V)} \mathbb{R} \cong A_{\mathbb{R},V}$ and $\mathbb{R}$ does not split the quaternionic extension of $Q(\chi_V)$ we must have that $A_{\mathbb{R},V}$ is a matrix algebra over the quaternions, hence $m_{\mathbb{R}}(V) = 2$. Clearly $m_Q(V) = 1$ implies $m_{\mathbb{R}}(V) = 1$.

Corollary 9.3.2. *Let G be a p-group. Then:*

i) *If* $p \neq 2$ *then* $R(G,Q) \cong R(G)^{\Gamma}$.

ii) *For arbitrary* p $R(G,Q) = RO(G)^{\Gamma}$.

Proposition 9.3.3. (Tornehave) *Let V be an irreducible complex representation of a 2-group G with* $\dim V^H$ *even for every subgroup H of G. Then V is quaternionic.*

Proof. (Tornehave) Let χ be the character of V and let $\mathrm{ind}_H^G 1_H$ be the character induced from the trivial character of H. Then by Frobenius reciprocity (Serre [147], 7.2) and the orthogonality relations

$$\langle \chi , \mathrm{Ind}_H^G 1_H \rangle = \dim V^H .$$

So the assumption on V means that χ has even multiplicity in every virtual permutation character. By Segal's theorem (section 4) we find that $\langle \chi , \xi \rangle$ is even whenever ξ is the character of a $Q[G]$-module. There is a unique irreducible $Q[G]$-module whose character ξ satisfies $\langle \chi , \xi \rangle \neq 0$. The even integer $m = \langle \chi , \xi \rangle$ is the Schur-index $m_Q(\chi)$. But $m_Q(\chi) = m_{\mathbb{R}}(\chi)$, and if this number is even V must be quaternionic.

9.4. The groups i(G) and iO(G)

The proof of the main theorem 9.1.6 will use induction over the order of the group. In this section we prepare this induction by presenting the relevant algebraic facts about i(G) and iO(G), in particular for p-groups.

For each orbit $A \in X = \mathrm{Irr}(G,\mathbb{C})/\Gamma$ we let F(A) be the free abelian group on its element. Then (additively) $R(G) = \oplus_{A\in X} F(A)$ and if we put $F_o(A) = R_o(G) \cap F(A)$ then $R_o(G) = \oplus_{A\in X} F_o(A)$. Moreover

$$F_o(A) = \left\{ \textstyle\sum_{a\in A} n_a a \;\middle|\; \sum n_a = 0 \right\}.$$

Since Γ is abelian the isotropy group of the Γ-action on A at $a \in A$ is independent of $a \in A$. Therefore we call this isotropy group Γ_A. We put $F_1(A) = I(\Gamma)F_o(A)$ and obtain $R_1(A) = \oplus_{A\in X} F_1(A)$ and

$$i(G) = \oplus_{A \in X} F_o(A)/F_1(A) .$$

The map

$$\Gamma / \Gamma_A \longrightarrow F_o(A)/F_1(A) \; : \; \gamma \longmapsto (1-\gamma)V$$

for $V \in A$ is independent of V and is seen to be an isomorphism. Thus we obtain a canonical isomorphism

(9.4.1) $$i(G) \cong \oplus_{A \in X} \Gamma / \Gamma_A$$

which we sometimes regard as an identification.

We need some functional properties of this map. The group $\Gamma = \Gamma(m)$ is not uniquely determined by G because m could be any multiple of $|G|$. If we are dealing with several groups we want m to be a multiple of all their orders. For a more functorial treatment one should use instead of Γ a profinite group, e. g. the Galois group of the field generated by all roots of unity over Q. This point of view is not so important for us. Nevertheless Γ / Γ_A is, by elementary Galois-theory, in a canonical way independent of m.

The restriction of the group action to a subgroup H induces a homomorphism

$$res_H : i(G) \longrightarrow i(H) .$$

We need a description of res_H in terms of the isomorphism 9.4.1. If $V \in A \in X(G)$ then $res_H V$ splits into irreducible H-modules, say

$$\mathrm{res}_H\, V = \bigoplus_{i=1}^{t} \left(\bigoplus_{j=1}^{n(t)} W_{ij} \right)$$

where the index i collects all those summands which belong to the same Γ-orbit, A(i) say, of Irr(H,$\mathbb{C}$). Then res_H is the direct sum of the maps

(9.4.2)
$$\Gamma / \Gamma_A \longrightarrow \bigoplus_{i=1}^{t} \Gamma / \Gamma_{A(i)}$$

$$\gamma \longmapsto (\gamma^{n(1)}, \ldots, \gamma^{n(t)})$$

This is easy to verify.

The computation of i(G) above can be done in a completely analogous manner for iO(G). We obtain an isomorphism as in 9.4.1.

We now come to another description of i(G) and iO(G), valid for p-groups. We need an elementary Lemma. Let a cyclic group Γ act on a free abelian group A as a group of automorphism. Let $\gamma_o \in \Gamma$ be a generator of this group. Put $A_\Gamma = A/(1-\gamma_o)A$, $(1-\gamma_o)^i A_{i-1}$ for $i \geq 1$, $i(A) = A_o/A_1$.

<u>Lemma 9.4.3. The following sequence is exact</u>

$$0 \longrightarrow A^\Gamma \longrightarrow A_\Gamma \xrightarrow[1-\gamma_o]{} i(A) \longrightarrow 0 .$$

<u>Proof.</u> Suppose $a \in A^\Gamma$ maps to zero in A_Γ. Then $a = (1-\gamma_o)b$ and therefore $|\Gamma| a = \sum_{\gamma \in \Gamma} \gamma a = \sum \gamma(1-\gamma_o)b = 0$. Since A is free we must have a = o, hence the map $A^\Gamma \longrightarrow A_\Gamma$ is injective. By definition $A_\Gamma \longrightarrow i(A)$ is surjective (and well-defined). If a is in the kernel of this map then $(1-\gamma_o)a = (1-\gamma_o)^2 b$ and therefore the element

$c = a-(1-\gamma_o)b$, which represents the same element as a in A_{Γ}, satisfies $c = \gamma_o c$ and therefore lies in A^{Γ} because γ_o is a generator.

Now we note that our group Γ can be taken to be cyclic if G is a p-group ($p \neq 2$) and $\Gamma / \{\pm 1\}$ is also cyclic for p = 2. Therefore the Lemma yields

<u>Proposition 9.4.4.</u> <u>Let</u> G <u>be</u> <u>a</u> p-<u>group</u>. <u>The</u> <u>following</u> <u>sequences</u> <u>are</u> <u>exact</u>:

(1) $$0 \longrightarrow RO(G)^{\Gamma} \longrightarrow RO(G)_{\Gamma} \xrightarrow[s_k]{} iO(G) \longrightarrow 0$$

<u>and similar sequences with</u> RO <u>replaced by</u> RSO <u>or the augmentation ideals</u> IO <u>and</u> ISO.

(2) (For $p \neq 2$)

$$0 \longrightarrow R(G)^{\Gamma} \longrightarrow R(G)_{\Gamma} \xrightarrow[s_k]{} i(G) \longrightarrow 0$$

<u>and similarly for the augmentation ideal</u> I(G) <u>instead of</u> R(G).

For the rest of this section G will be a p-group.

Let V be an irreducible G-module with kernel H. We call V <u>primitive</u> if G/H is a cyclic, dihedral, or generalized quaternion group, and imprimitive otherwise. Let X'(G) be the set of Γ-orbits of imprimitive G-modules. Let i'(G) be the subgroup of i(G) that corresponds to $\bigoplus_{A \in X'(G)} \Gamma / \Gamma_A$ under the isomorphism 9.4.1. We define analogously $iO'(G) \subset iO(G)$. The importance of the primitive modules comes from the following variant of Blichfeldt's theorem which we state for later use as a Lemma.

Let V be an irreducible complex G-module which is isomorphic to its dual V^*. Then there exists a conjugate linear map $J : V \longrightarrow V$ with either $J^2 = \mathrm{id}$ (V of real type) or $J^2 = -\mathrm{id}$ (V of quaternionic type).

Lemma 9.4.5. *An imprimitive G-module V of real (resp. quaternionic) type is induced from a real (resp. quaternionic) module of a proper subgroup.*

Proof. We give a proof in the quaternionic case. (The real case is analogous.) Assume that V as a quaternionic G-module is not induced from a proper subgroup. We may assume that V is faithful and want to show that G is cyclic or generalized quaternion, in this case. Let K be a maximal normal abelian subgroup of G. If the restriction $\mathrm{res}_K V$ would contain two non-isomorphic irreducible quaternionic modules then V would not be irreducible. (See Curtis-Reiner [48], § 49 - 50, and note that the considerations apply to quaternionic modules.) Therefore $\mathrm{res}_K V \cong V_o + \dots + V_o$ with some irreducible quaternionic K-module V_o. Since V_o is faithful and K is abelian we must have that K is cyclic and $\dim_{\mathbb{H}} V_o = 1$ ($\mathbb{H}$ = quaternions). Since K was a maximal abelian normal subgroup, G/K acts via conjugation faithfully on K. The module V_o is a complex K-module of the form $W_o \oplus W_o^*$. If $g \in G \setminus K$ and $k \in K$ is a generator then $gkg^{-1} \neq k$. Therefore conjugation by g interchanges W_o and W_o^* and acts as $gkg^{-1} = k^{-1}$ because V_o is a faithful K-module. This implies that the order of G/K is at most 2 and therefore that G is either cyclic (G = K) or dihedral or generalized quaternion. But a dihedral group has no quaternionic irreducible modules.

Let $\mathrm{res} : i(G) \longrightarrow \prod_H i(H)$ be the product of the restriction maps res_H where H runs through the maximal proper subgroups of G. We also let res be the restriction of this map to $i'(G)$. We have a similar map in the real case.

Proposition 9.4.6. *The map*

$$\mathrm{res} : iO'(G) \longrightarrow \prod_H iO(H)$$

is injective. The map

$$\mathrm{res} : i'(G) \longrightarrow \prod_H i(H)$$

is injective if G has odd order.

The rest of this section is concerned with the proof of this Proposition. The essential fact is isolated in Lemma 9.4.7 which implies the Proposition easily if we use the isomorphism 9.4.1 and the commutative diagram

$$\begin{array}{ccc} i'(G) & \xrightarrow{\mathrm{res}} & \prod_H i(H) \\ \downarrow \cong & & \downarrow \cong \\ \prod_{A \in X'(G)} \Gamma/\Gamma_A & \xrightarrow{\mathrm{res}} & \prod_H (\prod_{D \in X(H)} \Gamma/\Gamma_D) \end{array}$$

where the description of the bottom map is given in 9.4.2. Similarly in the real case.

Now suppose $x = (\gamma_A \in \Gamma/\Gamma_A \mid A \in X'(G))$ is given.

Lemma 9.4.8. *Assume* $p \neq 2$ *in the complex case. For each* $A \in X'(G)$ *there exists a maximal proper subgroup* H *of* G *and a* $C \in X(H)$ *such that the following holds:*

i) *For* $A \neq B \in X'(G)$ *the* C-*component of* $\mathrm{res}_H \gamma_B \in \prod_{D \in X(H)} \Gamma/\Gamma_D$ *is zero.*

ii) $: \Gamma / \Gamma_A \xrightarrow[\mathrm{res}_H]{} \prod_{D \in X(H)} \Gamma / \Gamma_D \xrightarrow[\mathrm{pr}_C]{} \Gamma / \Gamma_C$

is injective.

Proof. We begin with the complex case and allow also $p = 2$ in the following recollection of representation theory.

Let $V \in A \in X'(G)$. Since V is imprimitive we have $\dim_{\mathbb{C}} V > 1$. By the theorem of Blichfeldt (Serre [147], 8.5) we can find a proper subgroup H of G such that V is induced from an irreducible H-module W, notation: $V = \mathrm{ind}_H^G W$. By transitivity of induction we can moreover assume that H is a maximal proper subgroup of G. Then H is normal in G with index p. We choose H and $W \in C \in X(H)$ with these properties to prove the assertion of the Lemma.

We have a splitting $\mathrm{res}_H V \cong W_1 \oplus \ldots \oplus W_p$ with $W_1 = W$, say, and the W_i are pairwise non-isomorphic (Serre [147], 7.4). If U is irreducible and W is a direct summand of $\mathrm{res}_H U$, then by Frobenius reciprocity

$$0 \neq \langle \mathrm{res}_H U, W \rangle_H = \langle U, \mathrm{ind}_H^G W \rangle = \langle U, V \rangle$$

and hence $U \cong V$. This proves i). We note that $V \cong \mathrm{ind}_H^G W_i$. For the proof of ii) we consider several cases.

First case. The W_i belong to different Γ-orbits. Since induction is compatible with the Γ-action we obtain $\Gamma_C \subset \Gamma_A$. But if $\gamma \in \Gamma_V$ then

$$W_1 \oplus \ldots \oplus W_p \cong \gamma W_1 \oplus \ldots \oplus \gamma W_p$$

and therefore $\gamma W_i = W_i$ for all i because the W_i belong to different

Γ-orbits. Hence also $\Gamma_A \subset \Gamma_C$ and the map ϱ is the identity in this case.

Second case. There exists $\gamma_o \in \Gamma$ with $\gamma_o W_i \cong W_j$ for a pair $i \neq j$. Then $V = \gamma_o V$ and therefore $\gamma_o \in \Gamma_V$ permutes $W_1, \dots, W_p$. This has to be a cyclic permutation. Hence $\gamma_o^p \in \Gamma_C$ and Γ_A / Γ_C has exponent p. From 9.4.2 we see that ϱ is given by $\varrho(\gamma) = \gamma^p$. If p is odd, Γ is cyclic of order $(p-1)p^k$ for a suitable k and ϱ must be injective ($p \neq 2$).

If $p = 2$ then $\Gamma = Z/2 \times Z/2^k$ for a suitable k. If $Z/2 \subset \Gamma_A$ this means $V = V^*$. Then either $W_1 = W_1^*$, $W_2 = W_2^*$ or $W_1 = W_2^*$, $W_2 = W_1^*$. In the first case ϱ is still injective, reasoning as for $p \neq 2$. By 9.4.5 we can avoid the case $W_1 = W_2^*$. If Z/2 is not contained in Γ_A then this factor of Γ is contained in the kernel of ϱ.

We now turn to real G-modules. Then Γ / Γ_A is always cyclic. If $res_H V$ splits into p non-isomorphic irreducible real H-modules the same proof as above works. We look at the irreducible real G-modules according to their endomorphism ring which is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The cases End(V) = $\mathbb{R}$, $\mathbb{H}$ can only occur for 2-groups (Serre [147], p. 122).

End(V) = $\mathbb{C}$. Then V is obtained by restriction of scalars from a complex G-module U with $U \neq U^*$, notation: $rU = V$. Then

$$res_H V = res_H rU = r\, res_H U = rU_1 \oplus \dots \oplus rU_p .$$

A relation $U_i = U_j$ would imply $U = U^*$. Hence the $U_1, \dots, U_p, U_1^*, \dots, U_p^*$ are all distinct and therefore $rU_i = V_i$ are distinct real G-modules. If V_i is a direct summand in $res_H V'$ for an irreducible real G-module V' then Frobenius reciprocity again would imply that $V' = V$.

End(V) = $\mathbb{R}$. Then the complexification cV of V is irreducible. Since $\dim_{\mathbb{C}} V > 1$ we have $\mathrm{res}_H\, cV = W_1 \oplus W_2$ for a suitable subgroup H of index 2 in G. We must have $(W_1 \oplus W_2)^* = W_1 \oplus W_2$ and therefore $W_1 = W_1^*$, $W_2 = W_2^*$ or $W_1 = W_2^*$, $W_2 = W_1^*$. By 9.4.5 we can avoid the second case, hence we still have $W_i = cV_i$ with irreducible V_i and V_1, V_2 are not isomorphic.

End(V) = $\mathbb{H}$. Then V is obtained by restriction of scalars from an irreducible quaternionic G-module U, notation: rU = V. Again by 9.4.5 we can assume that $\mathrm{res}_H\, U$ splits into two non-isomorphic H-modules for suitable H and therefore $\mathrm{res}_H\, V$ splits into two non-isomorphic irreducible H-modules.

9.5. Construction of homotopy-equivalences.

We prove Theorem 9.1.4, namely the inclusions

$$R_1(G) \subset R_h(G), \quad RO_1(G) \subset RO_h(G).$$

We begin with an example due to Ted Petrie.

Let G be the cyclic group of order n with generator g. Let V^a be the $\mathbb{C}[G]$-module $\mathbb{C}$ with g acting as multiplication with $\exp(2\pi ia/n)$. Let a and b be integers, relatively prime and prime to n. Choose integers p,q such that $-ap + bq = 1$. The map

$$\text{(9.5.1)} \qquad f : V^a \oplus V^b \longrightarrow V^1 \oplus V^{ab}$$
$$(x,y) \longmapsto (x^p \bar{y}^q,\ x^b + y^a)$$

is a G-map. We claim that f has degree one. Consider the value (1,0).

It is easy to see that $f(x,y) = (1,0)$ implies $(x,y) = ((-1)^q, (-1)^p)$. One calculates the jacobian point to be $a^2p^2 + b^2q^2$. If this would be zero then we would obtain, using $-ap + bq = 1$, that $-2abpq = 1$ which is impossible because a,b,p,q are integers. Since f is a proper map it induces a map of degree one between the one-point compactifications. Also a G-map between unit spheres

$$h : S(V^a \oplus V^b) \longrightarrow S(V^1 \oplus V^{ab})$$

$$h(x,y) = f(x,y)/\| f(x,y)\|$$

is induced. We can see that h has degree one: The radial extension of h to a map $h_1 : V^a \oplus V^b \longrightarrow V^1 \oplus V^{ab}$ has the same degree as h, and h_1 is properly homotopic to f. Since h is a G-map between free G-spaces which is an ordinary homotopy equivalence, it is a G-homotopy equivalence by Proposition 8.2.1.

Now given $E-F \in R_1(G)$ for a cyclic group G. Then E-F is an integral linear combination of elements $(1-\psi^a)(1-\psi^b)U$ where a and b are prime to $|G|$. If $(a,b) = 1$ then the example of Petrie above shows that $(1-\psi^a)(1-\psi^b)U \in R_h(G)$ because we actually have constructed an oriented homotopy equivalence. If a and b are not relatively prime than we replace b by a suitable b+kn such that $(a,b+kn) = 1$. Hence we have shown that $R_1(G) \subset R_h(G)$ for cyclic G.

We use induced representations to prove the general result. If $H < G$ and $\mathrm{ind}_H^G : R(H) \longrightarrow R(G)$ is the homomorphism given by induced representations then

(9.5.2) $$\mathrm{ind}_H^G(R_h(H)) \subset R_h(G).$$

(9.5.3) $$\mathrm{ind}_H^G(R_i(H)) \subset R_i(G), \qquad i = 0,1.$$

The relation 9.5.3 follows from the fact that ind_H^G commutes with the Γ-action; and to prove 9.5.2 we note that

$$S(ind_H^G W) \cong *_{gH \in G/H} S(gH \times_H W),$$

so that homotopy equivalences for H-modules induce homotopy equivalences for the induced G-modules by taking suitable maps on the join. By the result above for cyclic G and 9.5.2 - 3 we see that $R_1(G) \subset R_h(G)$ whenever irreducible G-modules are induced from one-dimensional G-modules. This holds for p-groups and more generally for supersolvable groups (Serre [147], 8.5. Thèorème 16), and in particular for extensions of cyclic groups by p-groups. Now we can apply a general induction theorem of Dress [80] to conclude that $R_1(G) \subset R_h(G)$ for general G (see also section 6): The functors R_1 and R_h are compatible with restriction and induction (9.5.2 - 3). They are therefore sub-Mackey-functors of the representation ring functor. Therefore elements in $R_1(G)$ are induced from hyperelementary subgroups H of G (i. e. $0 \to S \to H \to P \to 0$, S cyclic, P a p-group). But for such groups H we know already that $R_1(H) \subset R_h(H)$. This proves Theorem 9.1.4 in the complex case.

In the real case we again need only consider groups G which are extensions of cyclic groups by p-groups. Using induction we reduce to the case of a real faithful irreducible G-module M which is not induced from a proper subgroup. The arguments of Dress [81] , p. 318, then show that either G is cyclic and $\dim_{\mathbb{R}} M \leq 2$ or G is dihedral and $\dim_{\mathbb{R}} M = 2$. If G is cyclic and $\dim_{\mathbb{R}} M = 1$ then (M being faithful) G = Z/2 and the Γ-action is trivial. If G is cyclic and $\dim_{\mathbb{R}} M = 2$ then M is obtained from a complex G-module by restriction of scalars. The restriction is compatible with the Γ-action, hence $(1-\gamma)(1-\sigma) M \in RO_h(G)$ follows in this case from the analogous statement for complex modules. If G is dihedral with generators g,t and relations $g^n = gtgt = t^2 = 1$ then the possible M have the form: M = $\mathbb{C}$, g acts through multiplication with $\exp(2\pi ij/n)$, $(j,n) = 1$, and t acts as complex conjugation. In this case 9.5.1 still works. This finishes the proof in the real case.

Remark. A different proof for Theorem 9.1.4 will be given in section 10. This proof uses the Galois invariance of certain stable homotopy modules over the Burnside ring.

9.6. Homotopy equivalences for p-groups.

We prove Theorem 9.1.6. This Theorem tells which representations of p-groups are (oriented) stably homotopy equivalent. The proof will be done by induction over the order of the group. Later we shall present a more conceptual proof which also gives better results. We assume in this section that 9.1.6 holds for cyclic, dihedral, and quaternionic groups; this is essentially classical (see de Rahm [132] ,) and will be reproved in 9.7 after we have developed some general facts from equivariant K-theory.

Let G be a p-group. Let S(G) be the set of normal subgroups of G. If a G-module V is given we write

$$V = \bigoplus_{H \in S(G)} V(H)$$

where V(H) collects the irreducible submodules of V which are lifted from faithful irreducible G/H-modules (i. e. have kernel H).

Lemma 9.6.1. If $x = V-W \in R_h(G)$ *(resp.* $RO_h(G)$*) then for all* $H \in S(G)$ *we have* $x(H) := V(H) - W(H) \in R_h(G)$ *(resp.* $RO_h(G)$*). (Here G can be an arbitrary group.)*

<u>Proof.</u> Let $f : S(V \oplus U) \longrightarrow S(W \oplus U)$ be a G-homotopy equivalence. If $H \in S(G)$ is a maximal proper subgroup of G (among the isotropy groups on V) then $S(V \oplus U)^H = S(V^G \oplus V(H) \oplus U^H)$ and therefore f^H gives a stable homotopy equivalence between V(H) and W(H), which is oriented if f was oriented. But because $R_h(G)$ is a subgroup of R(H) we can subtract x(H) from x and use the same argument for x - x(H). Downward induction over the $H \in S(G)$ gives the result.

We let j(K,f) be the j-group built from faithful irreducible K-modules, i. e. $j(K,f) = R_o(K,f)/R_h(K,f)$ where $R_o(K,f)$ is the set of x = V-W with V and W direct sums of faithful irreducible K-modules and $R_h(K,f)$ the subgroup of those $x = V-W \in R_o(K,f)$ such that V and W are oriented stably homotopy equivalent. We have similar groups i(K,f), iO(k,f), and jO(K,f). Lemma 9.6.1 tells us that we have a splitting

$$(9.6.2) \qquad s : j(G) \cong \prod_{H \in S(G)} j(G/H,f)$$

mapping x to $(x(H) | H \in S(G))$. The isomorphism 9.4.1 yields a similar splitting for i(G). The map t(G) is compatible with this splitting, it is therefore a direct sum of maps

$$t(G/H,f) : i(G/H,f) \longrightarrow j(G/H,f) .$$

It is enough to study the maps t(K,f) and similarly defined maps tO(K,f). They are surjective by definition. Our assumption in the beginning of this section was that these maps are injective if K is cyclic, or if K is a dihedral or generalized quaternion 2-group. By Proposition 9.4 and induction over the group order, t(G/H,f) and tO(G/H,f) is injective if we deal with imprimitive modules ($p \neq 2$ in the complex case). By 9.4 the possible kernel of t(G) for 2-groups G may be described as follows: It is generated by elements $V-V^*$, where V is an irreducible G-module with $V \neq V^*$ and dim $V^H \equiv 0 \bmod 2$ for all $H < G$. But by 9.3.3 this case cannot occur. This finishes the proof of 9.1.6.

9.7. Equivariant K-theory and fixed point degrees.

Let V and W be complex G-modules. Let $f : V^c \longrightarrow W^c$ be a pointed G-map between their one-point-compactifications. In this section G is a compact Lie group, if not otherwise specified. We apply equivariant complex K-theory to f and obtain an induced homomorphism

$$f^* : \tilde{K}_G(W^c) \longrightarrow \tilde{K}_G(V^c) .$$

By the equivariant Bott-isomorphism (Atiyah [10]) $\tilde{K}_G(V^c)$ is a free R(G)-module with generator $\lambda(V)$, the Bott class. Therefore f defines an element $z_f = z \in R(G)$ by $f^* \lambda(W) = z\lambda(V)$. We think of z being a character, i. e. a function on G. We want to compute this character.

Let $C < G$ be a topologically cyclic subgroup with generator g (i. e. powers of g are dense in C). Consider the following diagram (with $K_G(V)$ for $\tilde{K}_G(V^c)$)

$$\begin{array}{ccc} K_G(W) & \xrightarrow{\quad f^* \quad} & K_G(V) \\ \big\downarrow r & & \big\downarrow r \\ K_C(W^C) & \xrightarrow{\quad (f^C)^* \quad} & K_C(V^C) \end{array}$$

where the vertical maps are given by restriction to C and its fixed point sets. Since C acts trivially on V^C and W^C we have

$$(f^C)^* \lambda(W^C) = d(f^C) \lambda(V^C),$$

$d(f^C)$ = degree of f^C. We put $d(f^C) = 0$ if $\dim W^C \neq \dim V^C$. Moreover from elementary properties of Bott-classes we have

$$r\ \lambda(W) = \lambda_{-1}(W_C)\ \lambda(W^C)$$

where W_C is a complement of W^C in W (as C-module) and λ_{-1} is the alternating sum $\sum(-1)^i \lambda^i$ of the exterior powers. If we put this together we obtain

$$\lambda_{-1}(W_C)\ d(f^C) = \mathrm{res}_C z \cdot \lambda_{-1}(V_C). \tag{9.7.1}$$

If C is a torus we can solve for $\mathrm{res}_C z$ because R(C) has no zero-divisors. In general we evaluate characters at the generator $g \in C$, observing that $\lambda_{-1}(V_C)(g) \neq 0$. Therefore we obtain the following expression for the character z

Proposition 9.7.2. *The character* z_f *has values*

$$z_f(g) = d(f^C)\ \lambda_{-1}(W_C - V_C)(g)$$

where C *is the closed subgroup generated by* $g \in G$.

Remark 9.7.3. In particular the right hand side of the equation in 9.7.2 is a character of G. This is in general not obvious and gives conditions on the degrees $d(f^C)$. We exploit this fact in section 10.

Corollary 9.7.4. *If* $V - W \in R_h(G)$ *then*

$$g \longmapsto \lambda_{-1}(W_g - V_g)(g)$$

is a character of G. (*Here* $W_g := W_C$)

We shall see, especially in section 10, that 9.7.4 is a strong condition for V-W to lie in $R_h(G)$, but it is awkward to work with and therefore we derive a simpler criterion using the Θ_k-operations of section 3. Namely if $k \in Z$ and $W = \psi^k V$ then we have

<u>Proposition 9.7.5. The function</u>

$$u(g) = k^{\dim V^g} \quad \lambda_{-1}(W_g - V_g)(g)$$

<u>is a character of</u> G, <u>namely the character of</u> $\Theta_k(V)$.

<u>Proposition 9.7.6. If</u> V <u>and</u> $\psi^k V$ <u>are oriented stably homotopy equivalent then</u>

$$e : g \longmapsto k^{\dim V^g}$$

<u>ia a character of</u> G.

<u>Proof. 9.7.4 and 9.7.5.</u>

We use the last Proposition to do some explicit calculations. Namely we prove the results missing in 9.6.

<u>Proposition 9.7.7. The maps</u> t(K,f) <u>and</u> tO(K,f) <u>are injective if</u> K <u>is an arbitrary cyclic group, or if</u> K <u>is a dihedral or generalized quaternion</u> 2-<u>group</u>.

<u>Proof. Cyclic groups</u>. Let K be the cyclic group of order n with generator g. Let V be the standard irreducible K-module with g acting as multiplication with $u_n = \exp(2\pi i/n)$. We have $i(K,f) \cong Z/n^*$, V- $\psi^k V$ corresponding to k mod n. Injectivity of t(K,f) means in this case:

$V - \psi^k V \in R_h(K)$ if and only if $k \equiv 1 \bmod n$. Proposition 9.7.6 says in this case: $e(1) = k$, $e(x) = 1$ for $x \neq 1$ is a character of K. For any character e of a group G we have $|G|^{-1} \sum_{x\in G} e(x) \in Z$ because this is the multiplicity of the trivial character in e. Hence $\sum_{x\in G} e(x) \equiv 0 \bmod |G|$ In our case this yields $k+(n-1) \equiv 0 \bmod n$, i. e. $k \equiv 1 \bmod n$ as was to be shown.

In the case of real representations we allow also degrees -1. Hence we have to see wether $e(1) = k$, $e(x) = -1$ for $x \neq 1$ defines a character of G. This gives $k \equiv -1 \bmod n$, in accordance with $iO(K,f) = (Z/n)^*/\{\pm 1\}$.

Generalized quaternion groups. Let K be the group of order 2^{n+1} given by generators A,B and relations $BAB^{-1} = A^{-1}$, $A^{2^{n-1}} = B^2$, $n \geq 2$. The faithful irreducible representations of K are given as follows. We put $m = 2^n$.

$$V_k(A) = \begin{pmatrix} u_m^k & 0 \\ 0 & u_m^{-k} \end{pmatrix}, \quad V_k(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $1 \leq k \leq 2^{n-1} - 1$ and $k \equiv 1 \bmod 2$. One has $\psi^k V_1 = V_k$. Moreover

$i(K,f) \cong (Z/m)^*/\{\pm 1\}$, $V_1 - V_k \longmapsto k \bmod m$. Proposition 9.7.6 says that $e(1) = k^2$, $e(x) = 1$ for $x \neq 1$, shall be a character of K if $V_1 - V_k \in R_h$. This implies $k^2 + (2^{n+1} - 1) \equiv 0 \bmod 2^{n+1}$ and hence $k \equiv \pm 1 \bmod m$, q. e. d.

In the real case the only new condition to be considered is $k^2 \equiv -1 \bmod 2^{n+1}$ which is impossible. Restriction of scalars defines an isomorphism $i(K,f) = iO(K,f)$ and $tO(K,f)$ is injective.

Dihedral groups. Let K be the group of order 2^{n+1} with generators A,B and relations $A^{2^n} = ABAB = B^2 = 1$. The faithful irreducible representations are given as follows. We put $m = 2^n$.

$$V_k(A) = \begin{pmatrix} \cos 2\pi k/m & -\sin 2\pi k/m \\ \sin 2\pi k/m & \cos 2\pi k/m \end{pmatrix}$$

$$V_k(B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $1 \leq k \leq 2^{n-1} -1$ and $k \equiv 1 \bmod 2$. We have $\psi^k V_1 = V_k$ and $i(K,f) \cong (Z/m)^*/\{\pm 1\}$. Proposition 9.7.6 says that $e(1) = k^2$, $e(A^i) = 1$ for $1 \leq i < m$, must be a character if $V_1 - V_k \in R_h$. One obtains $k^2+(m-1) + km \equiv 0 \bmod 2m$. This gives mod m $k \equiv \pm 1, \pm 1 + 2^{n-1}$ and only $k \equiv \pm 1$ lifts to a solution mod 2m. Whence injectivity of t(K,f).

Since the faithful irreducible real K-modules have no complex structure we use an ad hoc argument. The restriction to the cyclic subgroup C generated by A induces an isomorphism iO(H) = iO(C). But tO(C) is injective.

9.8. Exercises

1. Show that the functors $G \longmapsto j(G)$, $G \longmapsto jO(G)$ are modules over the Green functor "rational representation ring". Deduce that they satisfy hyperelementary induction.

2. Let V,W be complex G-modules which are oriented stably homotopy equivalent. Show that they are oriented homotopy equivalent. (Does an analogous assertion hold for real modules?)

3. Show by an example that $R_1(G) = R_h(G)$ is in general not true for non-p-group.

10. Geometric Modules over the Burnside Ring.

We investigate in this section stable equivariant homotopy sets of spheres. We consider them as modules over the Burnside ring using the fact the Burnside ring is isomorphic to stable equivariant homotopy of spheres in dimension zero. In order not to become involved in the homotopy group of spheres we mainly study those questions which only involve the concept of mapping degree. In particular we continue our study of homotopy equivalences between representations.

10.1. Local J-groups.

In order to prepare for the general study of vector bundles we study a somewhat weaker equivalence between representations than homotopy equivalence. In particular we recover results of Atiyah-Tall [14] , Lee-Wasserman [110] , Snaith [151] .

We call real G-modules V and W locally J-equivalent, in symbols $V \sim_{loc} W$, if for each subgroup $H < G$ there exists a G-module U and G-maps

$$f : S(V \oplus U) \longrightarrow S(W \oplus U), \quad g : S(W \oplus U) \longrightarrow S(V \oplus U)$$

such that f^H and g^H have degree one. (Note that these degrees depend on the choice of orientations and are therefore only defined up to sign.) We put

$$(10.1.1) \qquad TO_G = \{ V-W \in RO(G) \mid V \sim_{loc} W \}$$

$$JO_G^{loc} = RO(G)/TO_G.$$

Note that we have a canonical quotient map

$$q(G) : JO_G^{loc} \longrightarrow RO(G)/RO_o(G) =: RO(G)_\Gamma ,$$

provided G is a finite group.

Theorem 10.1.2. *For every finite group* G *the map* q(G) *is an isomorphism.*

Proof. We have to show that for a G-module V and k prime to $|G|$ the relation $V \sim_{loc} \psi^k V$ holds. We can assume that k is an odd integer. We first show that there exist stable maps $f : V \longrightarrow \psi^k V$ such that for all $H < G$ the degree of f^H has the form k^t. (A stable map $f : V \longrightarrow W$ is a map $f : S(V \oplus U) \longrightarrow S(W \oplus U)$ for suitable U). If V is one-dimensional there is no problem. Next suppose that V is two-dimensional and irreducible; then G/ker V =: K is cyclic or dihedral and $V = \mathbb{C}$ with suitable action (see 9.7) and the map f can be taken as $z \longmapsto z^k$. In general, by a theorem of Brauer (Serre [147] , 12.6), we can write $V = \Sigma\, n_i \operatorname{ind}_{H_i}^G V_i$, $n_i \in Z$, V_i irreducible of dimension ≤ 2. Since G is prime to k induction commutes with ψ^k. Hence we have stable maps $\operatorname{ind}_{H_i}^G V_i \longrightarrow \operatorname{ind}_{H_i}^G \psi^k V_i$ of the required type. Moreover we can find an integer n such that $\psi^{k^n} V_i = V_i$ (choose n so that $k^n \equiv 1 \bmod |G|$). Hence we can find stable maps $\operatorname{ind}_{H_i}^G \psi^{k^{n-1}} V_i \longrightarrow \operatorname{ind}_{H_i}^G V_i$ so that negative n_i in the expression for V don't make trouble. Since we can find numbers k and l with $(k,l) = 1$ and $\psi^k V = \psi^l V$ suitable linear combinations of stable maps f,g with degrees $d(f^H) = k^t$, $d(g^H) = l^u$ give a map h with $d(h^H) = 1$; q. e. d.

10.2. Projective modules.

We recall some of the homotopy notions introduced in section 8. Let E and F be real G-modules, G being a compact Lie group. Put $\alpha = E-F \in RO(G)$ and let

$$\omega_\alpha = \omega_\alpha^G = \{ S^E, S^F \}$$

be the stable G-equivariant homotopy group of pointed stable G-maps $S^E \longrightarrow S^F$. Here S^E denotes the one-point compactification of E. The groups ω_α are the coefficient groups of an equivariant homology theory. When we need space for lower indices we write

$$\omega_\alpha = \omega^{-\alpha}$$

Smashed product of representatives induces a bilinear pairing

$$\omega_\alpha \times \omega_\beta \longrightarrow \omega_{\alpha+\beta} .$$

In particular ω_α is a module over ω_o, the stable equivariant homotopy ring of spheres in dimension zero. The pairing above induces a homomorphism

(10.2.1) $$m_{\alpha,\beta} : \omega_\alpha \otimes_{\omega_o} \omega_\beta \longrightarrow \omega_{\alpha+\beta} .$$

Remark 10.2.2. The modules ω_α are determined by α only up to non-canonical isomorphism because in general S^E has many homotopy classes of equivariant self-homotopy-equivalences. This causes difficulties if one has to use associativity or commutativity of the pairing $m_{\alpha,\beta}$. A way out of these difficulties is to choose canonical representatives α = E-F or extra structure (like suitable orientations).

Theorem 10.2.2. Let α = E-F be in $RO_o(G)$ see (9.1). Then the following holds:

(i) The module ω_α is a projective ω_o-module of rank one.

(ii) For each $\beta \in RO(G)$ the pairing (10.1.1)

$$\omega_\alpha \otimes_{\omega_o} \omega_\beta \longrightarrow \omega_{\alpha+\beta}$$

is an isomorphism.

(iii) <u>The</u> ω_o-<u>module</u> ω_α <u>is free if and only if</u> E <u>and</u> F <u>are stably</u> G-<u>homotopy equivalent (in the sense of</u> 9.1).

We split the proof into a sequences of Propositions. The whole section is concerned with the proof.

First recall the definition (and result): Let P be a module over the commutative ring R. Then P is a projective R-module of rank one if and only if P is finitely generated and for each maximal ideal q of R the localization P_q at q is a free R_q-module of rank one (see Bourbaki [33], § 5 Théorème 2).

In the following we write

$$\omega = \omega_o, \quad \omega_\alpha = \omega^{-\alpha} .$$

We have shown in section 8 that ω is canonically isomorphic to the Burnside ring A(G). Using this isomorphism and the determination of the prime ideals of A(G) in 5.7 we can say:

Let $q \subset \omega$ be a maximal ideal. Then there exists a group $H < G$ (unique up to conjugation) such that NH/H is finite, the order of NH/H is prime to the characteristic $p \neq 0$ of ω/q and q is the kernel of mapping degree homomorphism d_H mod p where

(10.2.3)
$$d_H : \omega \longrightarrow Z$$
$$d_H [f] = \text{degree } f^H .$$

The corresponding ideal is then denoted q(H,p).

To define the mapping degree between different manifolds we need to choose orientations. Given E and F we choose orientations for S^E and S^F and define

(10.2.4) $$d_{\alpha,H} = d_H : \omega_\alpha \longrightarrow Z$$

by $d_{\alpha,H}[f]$ = degree f^H if dim E^H = dim F^H and = 0 otherwise. Then we show

<u>Proposition 10.2.5.</u> <u>If</u> $\alpha = E-F \in RO_o(G)$ <u>then there exists for each</u> $H < G$ <u>with</u> NH/H <u>finite and</u> $|NH/H| \not\equiv 0 \bmod p$ an $x \in \omega_\alpha$ <u>such that</u>

$$d_H\, x \not\equiv 0 \bmod p\ .$$

(Note that this assertion is independent of the ambiguity in the definition of d_H).

<u>Proof.</u> An algebraic proof for finite G is given in Theorem 10.1.2. We give a topological proof for general G. We first show the existence of an H-map $f : S^E \longrightarrow S^F$ such that f^H has degree one. (Since we are only interested in stable maps we can assume that dim E^H = dim $F^H > 1$.) By the assumption $\alpha \in RO_o(G)$ we have dim E^H = dim F^H and so we choose an H-map $f_1 : S^{E^H} \longrightarrow S^{F^H}$ of degree one. We extend f_1 to an H-map f using the obstruction theory of 8.3. The obstructions to extending over an orbit bundle lie in groups

$$H^i(X_n/G,\ X_{n-1}/G;\ \pi_{i-1}(Y^K))$$

where $X = S^E$, $Y = S^F$, $X_n \setminus X_{n-1} = X_{(K)}$ in an admissible filtration of X. Since $X_{(K)}/G = X_K/NK \subset X^K/NK$ and dim X^K = dim Y^K we see that the obstruction groups vanish for dimensional reasons. Hence an f exists as claimed. We now apply the transfer homomorphism

$$t_H^G : \omega_{\alpha|H}^H \longrightarrow \omega_\alpha^G$$

which satisfies

$$d_{\alpha,K}(t_H^G y) = \chi(G/H^K)\, d_{\alpha|H,K}(y) .$$

The element $x = t_H^G[f]$ has the desired property.

Proposition 10.2.6. *For* $q = q(H,p)$ *and* $\alpha \in RO_o(G)$ *the module* ω_q^α *is a free* ω_q*-module on one generator. The element* $x \in \omega_q^\alpha$ *is a generator if and only if* $d_{\alpha,H}(x) \not\equiv 0 \bmod p$.

Proof. Take $x \in \omega^\alpha$, $y \in \omega^{-\alpha}$. Then multiplication with x resp. y, using the pairing 10.2.1, gives ω-linear maps

$$y_* : \omega^\alpha \longrightarrow \omega \quad , \; x_* : \omega \longrightarrow \omega^\alpha$$

respectively. The composition $y_* x_*$ is multiplication with $yx \in \omega$. By definition of q(H,p) this element becomes a unit in ω_q if

$$d_H(yx) = \pm d_H(y)\, d_H(x) \not\equiv 0 \bmod p.$$

(Since d_H depends on the choice of orientations we have to put in a $\pm$.) A similar argument applies to $x_* y_*$. If xy is a unit in ω_q then x_{*q} is an isomorphism. By 10.2.5 we can find x,y such that xy becomes a unit in ω_q. This proves that ω_q^α is free with generator x. Since any other generator of ω_q^α differs from x by a unit of ω_q we also obtain the second assertion.

We now prove (ii) of the Theorem in case $\beta \in RO_o(G)$. Using a basic fact of commutative algebra (Bourbaki [33] , § 3.3.) we need only show that the localizations $(m_{\alpha,\beta})_q$ are isomorphisms, for each maximal ideal $q \subset \omega$. But then we are dealing with a map

$$\omega_q^{\alpha} \otimes \omega_q^{\beta} \longrightarrow \omega_q^{\alpha+\beta}$$

between free ω_q-modules of rank one (10.2.6), and the same Proposition tells us that the tensor product of the generators is mapped onto a generator.

We now finish the proof of (i) by showing

<u>Proposition 10.2.7.</u> <u>For</u> $\alpha \in RO_o(G)$ <u>the</u> ω-<u>module</u> ω_α <u>is finitely generated</u>.

<u>Proof.</u> By the remarks above we have an isomorphism $\omega_\alpha \otimes \omega_{-\alpha} \cong \omega$. Let the element $1 \in \omega$ correspond to $\sum_i m_i \otimes n_i$. Then ω_α is generated as ω-module by the m_i, namely for $x \in \omega_\alpha$

$$x = (\sum m_i \otimes n_i)x = \sum m_i(n_i x).$$

(This uses associativity of the pairings m).

<u>Remark 10.2.8.</u> If G is finite then $\omega_G^o(X;Y)$ is a finitely-generated ω_G^o-module if X and Y are finite G-CW-complexes. This follows by induction over the number of cells (using that ω_G^o is noetherian). What happens for G a compact Lie group?

In order to prove (ii) we note that an inverse to $m_{\alpha,\beta}$ is given by

$$\omega_{\alpha+\beta} \cong \omega_{\alpha} \otimes (\omega_{-\alpha} \otimes \omega_{\alpha+\beta}) \longrightarrow \omega_{\alpha} \otimes \omega_{\beta} .$$

Finally we show (iii). If E and F are stably G-homotopy equivalent then a stable equivalence induces an isomorphism $\omega_{\alpha} \cong \omega_{o}$. Conversely, assume that ω_{α} is free, with generator x say. Then $\omega_{-\alpha}$ is also free, because $\omega = \omega_{\alpha} \otimes \omega_{-\alpha} \cong \omega \otimes \omega_{-\alpha} \cong \omega_{-\alpha}$. Let y be a generator of $\omega_{-\alpha}$. The product $xy \in \omega$ is then a generator of this module, hence a unit of ω . This implies $d_H(xy) = \pm 1$ for all $H < G$ and therefore $d_H(x) = \pm 1$ for all $H < G$. By 8.2 x is represented by a G-homotopy equivalence.

<u>10.3. The Picard group and invertible modules.</u>

In order to use the results of 10.1 successfully we have to collect some facts about projective modules.

Let R be a commutative ring. The set of isomorphism classes of projective R-modules of rank one forms an abelian group under the composition law "tensor product over R". This group is called the <u>Picard group of</u> R

$$\mathrm{Pic}(R).$$

The inverse of an element is given by the dual module. Using the notations of section 9, part of 10.2.2 may be restated as follows

<u>Proposition 10.3.1. The assignment</u> $\alpha \longrightarrow \omega_{\alpha}$ <u>induces an injective</u> ring homomorphism

$$pO(G) : RO_o(G)/RO_h(G) \longrightarrow \mathrm{Pic}(\omega_o^G).$$

We are interested in the computation of Pic(ω_0^G) and pO(G). Since the results are interesting mainly for finite groups we assume from now on in this section that G is finite. This has the advantage that we can think of ω_0^G as a subring of a finite direct product of the integers.

The computation of Picard groups is facilitated by using the *Mayer-Vietoris sequence for* Pic.

Proposition 10.3.2. Let

$$\begin{array}{ccc} R & \xrightarrow{r_1} & R_1 \\ \Big\downarrow{\scriptstyle r_2} & & \Big\downarrow{\scriptstyle p_2} \\ R_2 & \xrightarrow{p_2} & S \end{array}$$

be a pull-back diagram of commutative rings.

Suppose that p_1 *is surjective. Then the following Mayer-Vietoris sequence is exact*

$$\begin{array}{l} \text{Pic } S \longleftarrow \text{Pic } R_1 \oplus \text{Pic } R_2 \longleftarrow \text{Pic } R \xleftarrow{d} S^* \\ \qquad \longleftarrow R_1^* \oplus R_2^* \longleftarrow R^* . \end{array}$$

Here S^* denotes the units of the ring S. We describe the maps in this sequence. If $f : R \longrightarrow S$ is a ring homomorphism we use f to view S as an R-module; if P is a projective R-module of rank one then $f_* P := P \otimes_R S$ is a projective S-module of rank one. The first two maps are given by $x \longmapsto (r_{1*} x, r_{2*} x)$ and $(y,z) \longmapsto p_{1*} y - p_{2*} y$ (consider Pic as additive group). The last two maps are given by similar formulas.

Now as to d. Given $e \in S^*$ let $l_e : S \longrightarrow S$ be the left translation $s \longmapsto es$. Let $M(e)$ be defined by the following pull-back diagram

(10.3.3)
$$\begin{array}{ccc} M(e) & \longrightarrow & R_1 \\ \big\downarrow & & \big\downarrow {\scriptstyle l_e p_1} \\ R_2 & \underset{p_2}{\longrightarrow} & S \end{array} \quad .$$

Then $M(e)$ is an R-module (an R-submodule of $R_1 \times R_2$). We need the following information about such modules. (We still assume the hypothesis of 10.3.2.)

<u>Proposition 10.3.4.</u> (i) $M(e_1) \oplus M(e_2) \cong M(e_1e_2) \oplus R$.

(ii) $M(e_1) \otimes M(e_2) \cong M(e_1e_2)$.

(iii) $M(e)$ <u>is projective of rank one</u>.

<u>Proof.</u> (i) The modules in question are given by the following pull-back diagrams

$$\begin{array}{ccc} M(e_1) \oplus M(e_2) & \longrightarrow & R_1 \oplus R_1 \\ \big\downarrow & & \big\downarrow {\scriptstyle h(p_1 x p_1)} \\ R_2 \oplus R_2 & \underset{p_2 \times p_2}{\longrightarrow} & S \oplus S \end{array} \qquad \begin{array}{ccc} M(e_1e_2) \oplus R & \longrightarrow & R_1 \oplus R_1 \\ \big\downarrow & & \big\downarrow {\scriptstyle k(p_1 x p_1)} \\ R_2 \oplus R_2 & \underset{p_2 \times p_2}{\longrightarrow} & S \oplus S \end{array}$$

where h is given by the matrix $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ and k by the matrix

$\begin{pmatrix} e_1e_2 & 0 \\ 0 & 1 \end{pmatrix}$. Now h and k differ by the matrix

$\begin{pmatrix} e_2 & 0 \\ 0 & e_2^{-1} \end{pmatrix}$ which can be lifted to an invertible matrix over R_1 because p_1 is surjective; here one uses the formal identity

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence $h(p_1 \times p_2)$ is transformable into $k(p_1 \times p_1)$ by invertible matrices so that (by transitivity of pull-backs) the desired isomorphisms drops out.

(iii) We obtain from (i) that $M(e) \oplus M(e^{-1})$ is free. Hence $M(e)$ is projective. If we localize (10.3.3) at prime ideals q of R we see that $M(e)_q = 0$ hence $\operatorname{rank}_q M(e) \geqslant 1$. Since $\operatorname{rank}_q M(e) + \operatorname{rank}_q M(e) = \operatorname{rank}_q (M(e) \oplus M(e^{-1})) = 2$, by (i), we have $\operatorname{rank}_q M(e) = 1$.

(ii) Since $M(e)$ has rank one the second exterior power Λ^2 of $M(e)$ is zero. Now apply Λ^2 to (i) and (ii) drops out.

If view of 10.3.4 we can now define a homomorphism

$$d : S^* \longrightarrow \operatorname{Pic} R \quad \text{by} \quad d(e) = M(e) .$$

With these preparations 10.3.2 is easy to verify.

The Mayer-Vietoris sequence may be applied to the Burnside ring A as follows. Let c be a multiple of the group order $|G|$. Let

$$\varphi : A = A(G) \longrightarrow C = C(\Phi(G), Z)$$

be the standard map. Then the following diagram is a pull-back

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & C \\ \downarrow & & \downarrow \\ A/cC & \xrightarrow[\varphi \bmod c]{} & C/cC \end{array} \quad .$$

Here the vertical maps are the canonical quotient maps. We regard φ as an inclusion. Since the cokernel of φ has exponent $|G|$ (section 1) we have $cC < A$ so that A/cC makes sense. We use the following facts.

Proposition 10.3.6. Pic $C = 0$. Pic $A/cC = 0$.

Proof. C is finite direct product of the integers, say $C = Z^n$. Since projective modules over Z are free we have Pic $Z = 0$. Using induction on n we obtain from 10.3.2 that Pic $Z^n = 0$.

In case of $R := A/cC$ we note that this ring is finite as an abelian group. Therefore R has a finite number of maximal ideals (is a *semi-local* ring). If $m_1,\dots,m_n$ are the maximal ideals then $R \longrightarrow \prod A/m_i$ is surjective (Chinese remainder theorem) with kernel $m = m_1 \cap \dots \cap m_n$ the radical. Since R is finite hence Artinian this radiacal equals the nil-radical nil R of R. The ring R/m is a product of fields hence Pic R/m is zero. We have proved Pic $R = 0$ if we use the following

Proposition 10.3.7. *Let I be an ideal in the commutative ring R. Then the canonical map*

$$\text{Pic } R \longrightarrow \text{Pic } R/I$$

is injective if I is contained in the radical of R and bijective if I is nilpotent.

Proof. The first statement follows from Bourbaki [33] , II § 3.2. Prop. 5. Now assume that I is nilpotent. We have to show that the map is surjective. A projective R/I-module of rank one is given as a direct summand of a finitely-generated free R/I-module hence is given by a certain idempotent matrix $A \in GL(n,R/I)$. We have to lift the matrix to an idempotent matrix $B \in GL(n,R)$. Once this is done the proof is finished because $\otimes_R R/I$ does not change the rank of a projective module. We define inductively a sequence of matrices as follows: Let $B_1 \in GL(n,R)$ be a lifting of A. Put $N_i = B_i^2 - B_i$ and $B_{i+1} = B_i + N_i - 2B_iN_i$. Then one checks that $N_{i+1} \in GL(n,I^{2^i})$ and that B_i is a lifting of A. For large i $N_i = 0$ and we are done.

Combining the previous results we obtain

Proposition 10.3.8. The following sequence is exact

$$0 \leftarrow \text{Pic } A \leftarrow (C/cC)^* \leftarrow C^* \oplus A/cC^* \leftarrow A^* .$$

In principal this sequence can be used to compute Pic A for the Burnside ring A = A(G). But it is not easy to obtain the actual structure of the abelian group Pic A. We shall indicate later, how the congruences 1.3 for the Burnside ring can lead to a computation.

Remark 10.3.9. If G is a compact Lie group Proposition 10.3.8 is still valid with c being a common multiple of the $|NH/H|$, $(H) \in \Phi(G)$. (See 5. 9. for the existence of such c.) One has the pull-back 10.3.5 and moreover Proposition 10.3.6. is still true.

We now continue with a pull back diagram as 10.3.5 where $C = Z^n$, is an inclusion of maximal rank. We consider C as an A-module via this inclusion. If $M,N \subset C$ are A-submodules we define their product

(10.3.10) $$MN \subset C$$

to be the module generated by all elements mn, $m \in M$, $n \in N$. We call M *invertible* if their exists N such that $MN = A$. (This is not quite the standard notion, e.g. as in Bourbaki [33], § 5.6, but exactly what we need. Therefore one should investigate a more general situation comprising both notions of invertible modules.) Let

$$\mathrm{Inv}\ (A)$$

be the set of invertible A-modules.

Proposition 10.3.11. (i) Inv A *is an abelian group under the composition law* 10.3.10.

(ii) *Invertible modules are projective of rank one. Assigning to each invertible module its class in* Pic A *we obtain a surjective homomorphism*

$$\mathrm{cl} : \mathrm{Inv}\ (A) \longrightarrow \mathrm{Pic}\ (A).$$

(iii) *There exists a canonical exact sequence*

$$0 \longrightarrow A^{*} \longrightarrow C^{*} \longrightarrow \mathrm{Inv}\ (A) \xrightarrow[\mathrm{cl}]{} \mathrm{Pic}(A) \longrightarrow 0.$$

(iv) *There exists a canonical exact sequence*

$$0 \longrightarrow (A/cC)^{*} \longrightarrow (C/cC)^{*} \longrightarrow \mathrm{Inv}\ (A) \longrightarrow 0.$$

Proof. (i) follows directly from the definition of Inv (A) because the existence of inverses was required.

(ii) Suppose $MN = A$. Then

$$CM = CMC \supset CMN = CA = C$$

hence
$$CM = C \ .$$

Therefore $1 = \sum c_i m_i$ for suitable $c_i \in C$ and $m_i \in M$ and hence $c = \sum (cc_i)m_i$. But $cc_i \in A$ so that $c \in M$, hence $cA \subset M$. In particular $M \subset C$ is a subgroup of maximal rank with cokernel annihilated by c, and $M \otimes_Z Q \longrightarrow C \otimes_Z Q$ is an isomorphism.

If $1 = \sum m_i n_i$, $m_i \in M$, $n_i \in N$ then $f_i : M \longrightarrow A : m \longmapsto mn_i$ is A-linear and for each $x \in M$ we have $x = \sum f_i(x)m_i$. Therefore M is a finitely generated projective module. Let q be a maximal ideal of A. Then M_q is a free A_q-module. Since $M_q \otimes \mathbb{Q} \cong C_q \otimes \mathbb{Q}$ as $A_q \otimes Q$-modules M_q must have rank one.

Finally given a projective module of rank one M. By 10.3.8 this module is isomorphic to a module of type M(e), $e \in (C/cC)^*$. We give another description of this module. Let $e' \in C$ be a lifting of e. Then M(e) can be identified with

(10.3.12) $$M'(e') := \{x \in C \mid e'x \in A\} \ .$$

Choose $f' \in C$ such that $e'f' = 1 + c^2z$ for an $z \in C$. Then $f' \in M'(e')$, $e' \in M'(f')$ and $e'f' = 1 + c^2z \in M'(e')\, M'(f') \subset M'(e'f')$. But $c \in M'(e')$ and $cz \in M'(f')$ hence $c^2z \in M'(e')\, M'(f')$ hence $A \subset M'(e')\, M'(f')$. On the other hand $M'(e'f') = M'(1+c^2z) = M'(1) = A$. Therefore $M'(e')$ is invertible and cl is surjective. From 10.3.4 (ii) we see that cl is a homomorphism.

(iii) Suppose that $M \in \mathrm{Inv}(A)$ is free, with generator x say. If $MN = A$ we must have an identity of the form $1 = \sum (a_i x)n_i$, so that $x \in C^*$ and $M = M'(x)$. If $M'(x) = M'(y)$ for $x,y \in C^*$ then $x = ay$ for $a \in A$; hence $a \in A^*$.

(iv) Let $r : C \longrightarrow C/cC$ be the quotient map. Let $C' = r^{-1}(C/cC^*)$. If $r(e) = r(f)$ then $M'(e) = M'(f)$: Let $e = f+ch$. Then $x \in M'(e) \Rightarrow ex \in A \Rightarrow x(f+ch) \in A$. Since $cC \subset A$ we conclude that $xch \in A$ and therefore $xf \in A$, so that $M(e) \subset M(f)$. We can therefore define a map $(C/cC)^* \longrightarrow \mathrm{Inv}(A)$ by $r(e) \longmapsto M'(e)$. To show that this is a homomorphism we note that $M'(e)\, M'(f) \subset M'(ef)$ which follows from the definition. This is an inclusion of invertible modules. Thus we have to show that any such inclusion $M \subset N$ must be an equality. Let q be a maximal ideal of A. By the Cohen-Seidenberg theorem (Atiyah-Mac Donald [11] , 5.) there exists a ring homomorphism $\varphi : C \longrightarrow Z$ such that $q = \{a \in A \mid \varphi(a) \equiv 0 \bmod p\}$ for some prime p. Therefore $x \in M$ is a generator of the localized module M_q if and only if $\varphi(x) \not\equiv 0 \bmod p$. Therefore $M_q \subset N_q$ maps a generator onto a generator, hence is an isomorphism. By commutative algebra, $M \subset N$ is an isomorphism.

The exactness of the sequence (iv) is implied by (iii) and 10.3.8.

We now prove a recognition principle for invertible modules.

Proposition 10.3.13. _Let_ M _be_ _invertible_. _Suppose_ $e \in M$ _and_ $r(ef) = 1$. _Then_ $M = M'(f)$.

Proof. If $x \in M'(f)$ then $xf \in A$ and therefore $xef \in M$. Since $cC \subset M$ we obtain $x \in M$ hence $M'(f) \subset M$. By the previous proof this inclusion must be an equality.

We conclude with a geometric application. Let $\alpha \in R_o(G)$. The module ω_α is contained via the mapping degree of fixed point mappings in $C(\phi(G),Z) = C$, see 8. 5 . We use this inclusion as an identification.

Proposition 10.3.14. (i) _Let_ $\alpha \in R_o(G)$. _Then_ $\omega_\alpha \subset C$ _is_ _invertible_.

(ii) <u>The assignment</u> $\alpha \longmapsto \omega_\alpha$ <u>induces a homomorphism</u>

$$R_o(G) \longrightarrow \mathrm{Inv}(\omega_o^G).$$

(iii) <u>For</u> $\alpha \in R_o(G)$ <u>the module</u> ω_α <u>is equal to</u> ω_o <u>if and only if</u> $\alpha \in R_h(G)$.

<u>Proof.</u> (i) We know already that ω_α is projective of rank one (10.2.2), but not every such submodule of C is invertible. The pairing 10.2.1

$$\omega_\alpha \otimes \omega_{-\alpha} \longrightarrow \omega_o$$

shows, by passing to fixed point degrees, that

$$\omega_o \supset \omega_\alpha \omega_{-\alpha} \ni 1$$

so that $\omega_o = \omega_\alpha \omega_{-\alpha}$.

(ii) The pairing 10.2.1 also shows $\omega_\alpha \omega_\beta \subset \omega_{\alpha+\beta}$. This being an inclusion of invertible modules is an equality by the proof of 10.3.11. (iv).

(iii) If $\omega_\alpha = \omega_o$ then $1 \in \omega_\alpha$. A map representing 1 is an oriented stable homotopy equivalence. Conversely $1 \in \omega_\alpha$ implies $\omega_\alpha = \omega_o$, by 10.3.13.

We restate 10.3.14 as follows

<u>Proposition 10.3.15. The assignment</u> $\alpha \longmapsto \omega_\alpha$ <u>induces an injective homomorphism</u>

$$p(G) : R_o(G)/R_h(G) \longrightarrow \mathrm{Inv}\,(\omega_o^G).$$

10.4 Comments.

This section is based on tom Dieck-Petrie [69], where further information may be found. Generalizations to real G-modules are in Tornehave [160]. A more conceptual proof of the main result of section 9 using section 10 and the theory of p-adic λ-rings may be found in tom Dieck [68]. These one also finds a computation of Pic A(G) for abelian G and an indication now Pic A(G) may be computed in general. For homotopy equivalent G-modules for compact Lie groups G see Traczyk [161]. For G-maps $S(V) \longrightarrow S(W)$ of specific degree see Lee-Wasserman [110] and Meyerhoff-Petrie [114]. An interesting and difficult problem is the study of homotopy equivalences between products $S(V) \times S(W)$. For the homeomorphism problem for the S(V) see Schultz [139].

11. Homotopy-equivalent stable G-vector bundles. *)

The aim of this section is to extend some of the previous results and techniques from representations to vector bundles. The group G will always denote a finite p-group and we are concerned with the question: When are the sphere bundles of two G-vector bundles stably G-fibre-homotopy equivalent?

11.1. Introduction and results about local J-groups.

One of the basic questions in the homotopy theory of vector bundles is the following: Given two vector bundles over a space X, when are the associated sphere bundles fibre-homotopy-equivalent?

The question has been answered, for stable bundles, by Adams in his series of papers on the groups J(X) [2], together with the affirmative solution of his famous conjecture (Quillen [128], Sullivan [157], Becker-Gottlieb [19]).

We shall extend some of these results to G-vector bundles. We consider G-vector bundles over finite G-CW-complexes. If $p : E \longrightarrow X$ is such a bundle we can choose a G-invariant Riemannian metric on E and consider the unit-sphere bundle $S(E) \longrightarrow X$. If V is a real G-module we also let V denote the product bundle $V \times X \longrightarrow X$. If $p_i : E_i \longrightarrow X$ are G-vector bundles a stable map $f : S(E_1) \longrightarrow S(E_2)$ shall be a fibrewise G-map $S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$ for some G-module V. Two G-vector bundles $p_i : E_i \longrightarrow X$ over X are called stably-homotopy-equivalent, notation $E_1 \sim E_2$, if for some G-module V there exists a G-fibre-homotopy-equivalence $f : S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$. If E and F are G-vector bundles over X then $S(E \oplus F)$ is G-homeomorphic over X to the fibrewise join $S(E) * S(F)$. Using this it is easy to see that $E_1 \sim E_2$, $F_1 \sim F_2$ implies $E_1 \oplus F_1 \sim E_2 \oplus F_2$. Let $KO_G(X)$ be the Grothendieck ring

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*) This section contains joint work with H. Hauschild.

of real G-vector bundles over X. Then the previous remark shows that

$$(11.1.1) \qquad TO_G(X) = \{ E_1 - E_2 \in KO_G(X) \mid E_1 \sim E_2 \}$$

is well-defined and an additive subgroup of $KO_G(X)$. We pose the problem: Describe $TO_G(X)$ as a subgroup of $KO_G(X)$. The solution uses the computation of the J-groups

$$(11.1.2) \qquad JO_G(X) = KO_G(X)/TO_G(X).$$

We now introduce some intermediate J-groups where homotopy-equivalence is replaced by weaker conditions. Note that a G-fibre-homotopy-equivalence $f : S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$ induces an ordinary fibre-homotopy-equivalence f^H for all H-fixed point bundles ($H < G$ a subgroup of G). We therefore consider the following local condition: Two G-vector bundles E and F are called <u>stably locally homotopy-equivalent</u>, notation $E \sim_{loc} F$, if for every $H < G$ there exists a G-module V and fibrewise G-maps $f : S(E \oplus V) \longrightarrow S(F \oplus V)$ and $g : S(F \oplus V) \longrightarrow S(E \oplus V)$ such that f^H and g^H are ordinary fibre-homotopy-equivalences. As before it is seen that

$$(11.1.3) \qquad TO_G^{loc}(X) = \{ E_1 - E_2 \in KO_G(X) \mid E_1 \sim_{loc} E_2 \}$$

is well-defined and an additive subgroup of $KO_G(X)$. We study this subgroup via a computation of

$$(11.1.4) \qquad JO_G^{loc}(X) = KO_G(X)/TO_G^{loc}(X).$$

The introduction of these local J-groups may seem artificial at first sight. We offer some justification. Obviously we have a surjective homomorphism $JO_G(X) \longrightarrow JO_G^{loc}(X)$. If X is a point one obtains from

Atiyah-Tall [14] and tom Dieck [67] that this map is not an isomorphism: For p-groups it measures the difference between G-homotopy-equivalence and G-maps of degree one. It turns out that a computation of 11.1.4 will yield the main part of 11.1.2. Moreover $JO_G^{loc}(X)$ is actually computable using the action of the Adams operations on $KO_G(X)$ in the same way as the non equivariant J-groups are computed. So also from this point of view 11.1.3 is just the correct object to consider.

We now state our results on the computation of the local J-groups 11.1.4. It is expedient to consider the localizations

$$(11.1.5) \qquad JO_G^{loc}(X)_q = KO_G(X)_q / TO_G^{loc}(X)_q$$

where the index q indicates that we have localized at the rational prime q.

Given q let $r(1),\ldots,r(n)$ be a set of integers (depending on q and p) generating the q-adic units (modulo ± 1 if $q = 2$) and generating the units $Z/|G|Z^*$ of the integers modulo $|G|$. If $q = p$ then we take $n = 1$ and $r = r(1) = 3$ if $p = 2$, and r a generator of Z/p^2Z^* if $p \neq 2$. Our main result is the

<u>Theorem 11.1.6.</u> <u>Let</u> G <u>be a finite</u> p-<u>group</u>. <u>Then</u> $TO_G^{loc}(X)_q$ <u>is generated as abelian group by elements of the form</u> $x - \psi^{r(i)}x$, $x \in KO_G(X)_q$ $i = 1,\ldots,n$, <u>where</u> ψ^r <u>denotes the</u> r-<u>th Adams operation</u>.

The proof naturally splits into two parts. First we consider the case $p = q$. Here we prove an equivariant analogue of the Adams conjecture by elementary methods. We use the device of Becker-Gottlieb [19] but apply it to the universal example: orthogonal representations. We thus generalize the method which Adams [2] used for two-dimensional bundles. Moreover the main theorem of Atiyah-Tall [14] on p-adic λ-rings is used as well as the completion theorem of Atiyah-Segal [12]. The second part of the proof is essentially concerned with the situation where the order of the group is invertible. Here we can use the localization and splitting theorems of section 8 to decompose K-theory into simpler pieces for which the problem can easily be solved. We should point out that our exposition contains a computation of the non-equivariant J-groups which seems somewhat simpler than other published versions: We neither need Quillens computations nor infinite loop spaces.

11.2 Mapping degrees. Orientations.

This section contains some technical preparation. In particular we show that it suffices to consider orientable bundles.

An n-dimensional real G-vector bundle $E \longrightarrow X$ is called orientable if the n-th exterior power $\Lambda^n E$ is isomorphic to $X \times \mathbb{R} \longrightarrow X$ with trivial G-action on $\mathbb{R}$. Bundles E_1 and E_2 of dimension n are said to have the same orientation behaviour if $\Lambda^n E_1$ and $\Lambda^n E_2$ are isomorphic G-bundles. We put

$$(11.2.1) \qquad KSO_G(X) = \{ E_1 - E_2 \in KO_G(X) \mid E_i \text{ orientable} \} .$$

By a theorem of Dold [71] a fibrewise map $S(E) \longrightarrow S(F)$ is a fibre homotopy equivalence if and only if it is a homotopy equivalence on each fibre, i. e. has degree ± 1 on each fibre. It is therefore reasonable to ask for the existence of fibrewise G-maps with prescribed degree on the fibres.

Let $S \subset Z$ be a set of prime numbers. If E and F are G-vector bundles over X we write

(11.2.2) $$E_1 \leq_S E_2$$

if there exists a stable map $f : S(E) \longrightarrow S(F)$ with fibre degree prime to all elements of S. We write

(11.2.3) $$E \sim_S F \quad \text{if} \quad E \leq_S F \quad \text{and} \quad F \leq_S E.$$

We put

(11.2.4) $$TO_{G,S}(X) = \{E - F \in KO_G(X) \mid E \sim_S F\}$$

(11.2.5) $$JO_{G,S}(X) = KO_G(X)/TO_{G,S}(X) \ .$$

If S is the set of all primes then $E \sim_S F$ means that there exist stable maps $S(E) \longrightarrow S(F)$ and $S(F) \longrightarrow S(E)$ of degree ± 1 on the fibres.

<u>Lemma 11.2.6. Suppose there exists a fibrewise</u> G-<u>map</u> $f: S(E) \longrightarrow S(F)$ <u>of odd degree</u>. <u>Then</u>

$$E - F \in KSO_G(X).$$

<u>Proof.</u> Since Stiefel-Whitney classes are modulo 2 fibre-homotopy invariant we have $w_1(E) = w_1(F)$. If $w_1(E) \neq 0$ and $\Lambda^n E$ is the determinant bundle of E we have a fibrewise G-map $S(E \oplus \Lambda^n E) \longrightarrow S(F \oplus \Lambda^n E)$ of odd degree. We can therefore assume without loss of generality that E and F are orientable as bundles without group action. To show the determinant bundles are equal in this case we need only show that the G-action on each fibre is the same. But $g \in G$ acts as identity on the determinant bundle if it preserves the orientation and as minus identity otherwise and this distinction is preserved by a map of odd degree.

Corollary 11.2.7. $TO_G(X) \subset TO_G^{loc}(X) \subset KSO_G(X)$.

Let $B(G,O(1)) = B$ be the classifying space for one-dimensional G-bundles (tom Dieck). Then assigning to each bundle E its determinant bundle induces a split surjective homomorphism

$$\text{(11.2.8)} \qquad \det : KO_G(X) \longrightarrow [X,B]_G$$

with kernel $KSO_G(X)$; here $[-,-]_G$ denotes the set of G-homotopy classes. Using Corollary 11.2.7 we therefore obtain natural splittings

$$\text{(11.2.9)} \qquad JO_G(X) \cong JSO_G(X) \oplus [X,B]_G ,$$

with JSO = KSO/TO; and similarly for the local J-groups.

11.3. Maps between representations and vector bundles.

In this section we construct certain equivariant maps between orthogonal representations. The construction is a simple application of the methods in Becker-Gottlieb [19] and Meyerhoff-Petrie [114], and is essentially well known. These maps between representations will then give us maps between vector bundles.

Proposition 11.3.1. Let $\mathbb{R}^{2n}$ be the standard $O(2n)$-representation. Let k be a positive integer. Then there exist stable $O(2n)$-maps $S(\mathbb{R}^{2n}) \longrightarrow S(\psi^k \mathbb{R}^{2n})$ with degree a divisor of k^t for some $t \in \mathbb{N}$ if k is odd. (Otherwise for $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$.)

Remark. $\psi^k \mathbb{R}^{2n}$ may be a virtual $O(2n)$-module $V - W$, of course. The Proposition has to be read that there exists stable $O(2n)$-maps

$S(\mathbb{R}^{2n} \oplus W) \longrightarrow S(V)$. We use similar notations for vector bundles.

Proof. Let $T < O(2n)$ be a maximal torus with normalizer NT. Then $NT = S_n \times_s O(2)^n$, where S_n is the symmetric group and $\times_s$ means semi-direct product with respect to the permutation action of S_n on $O(2)^n$. We first show the existence of an NT-map of the required degree. Let

$$H = \{(s;x_1,\ldots,x_n) \in S_n \times_s O(2)^n \mid s(1) = 1\} .$$

One obtains a homomorphism

$$h : H \longrightarrow O(2) : (s;x_1,\ldots,x_n) \longmapsto x_1$$

and an associated 2-dimensional H-module V. The group H has finite index in NT, namely $[NT : H] = n$. Therefore one can consider induced representations ind_H^{NT}. One has

(11.3.2) $$\mathrm{ind}_H^{NT}\, V \cong W$$

where W is the standard NT-module (restriction of the standard O(2n)-module). See Becker-Gottlieb [19] for a proof of 11.3.2. If k is odd there is an O(2)-map $g : S(V) \longrightarrow S(\psi^k V)$; if $V = \mathbb{C}$ this is simply the map $z \longmapsto z^k$ (see Adams [2]). If k is even then $\psi^k(V) = \mu_k - \lambda_2 + 1$, where λ_2 is the determinant representation associated to the standard O(2)-action on $\mathbb{R}^2$ and where μ_k is $\mathbb{C}$ with $z \in S^1 = SO(2)$ acting as multiplication by z^k and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ acting as conjugation. There exists an O(2)-map $g' : S(V) \longrightarrow S(\mu_k)$, the map $z \longmapsto z^k$ as before. Since λ_2 and $\mathbb{R}$ have different orientation behaviour there does not exist a stable Z/2-map $S(\lambda_2) \longrightarrow S(\mathbb{R}^1)$. But $\lambda_2 \oplus \lambda_2$ and $\mathbb{R} \oplus \mathbb{R}$ have the same orientation behaviour and therefore we can find a stable Z/2-map (and hence O(2)-map)

$S(\lambda_2 \oplus \lambda_2) \longrightarrow S(\mathbb{R} \oplus \mathbb{R})$ of degree 2. Put together we see that there exists a stable H-map $g : S(V \oplus V) \longrightarrow S(\Psi^k(V \oplus V))$ whose degree divides some power of k.

Induction ind_H^{NT} yields a stable G-map

$$(11.3.3) \qquad \mathrm{ind}_H^{NT}(g) : S(\mathrm{ind}_H^{NT} V) = S(W) \longrightarrow S(\mathrm{ind}_H^{NT} \Psi^k V).$$

In order to finish the proof we need a stable NT-map

$$(11.3.4) \qquad h : S(\mathrm{ind}_H^{NT} \Psi^k V) \longrightarrow S(\Psi^k(\mathrm{ind}_H^{NT} V))$$

of suitable degree. For a prime p let $(NT/T)_p$ be the Sylow-p-group of NT/T and N_pT its counter-image in NT. If p is prime to k then $\mathrm{ind}_H^{NT}(\Psi^k V)$ and $\Psi^k(\mathrm{ind}_H^{NT} V)$ are isomorphic as N_pT-modules; this follows from two facts:

(11.3.5) If k is prime to the index $[G : H]$ then in general $\Psi^k \mathrm{in}_H^G = \mathrm{ind}_H^G \Psi^k$.

(11.3.6) $\mathrm{res}_{N_pT}^{NT} \mathrm{ind}_H^{NT}$ is by the double coset formula of representation theory a direct sum with summands of the form $\mathrm{ind}_K^{N_pT} \mathrm{res}_K^H$; and since $T < K$ the index $[N_pT : K]$ is prime to k.

Using this isomorphism of N_pT-modules we can find a stable NT-map h_p in 11.3.4 of degree $|NT/N_pT|$. Since the greatest common divisor of all the $|NT/N_pT|$ with p prime to k involves only prime divisors of k we can form a suitable linear combination of the h_p (in the homotopy group of stable maps) to produce an NT-map h whose degree divides a power of k.

As a consequence of Proposition 11.3.1 we obtain stable maps between vector bundles as follows. Let $E \longrightarrow B$ be a real G-vector bundle of dimension n (with Riemannian metric). The associated principal O(2n)-bundle $P \longrightarrow B$ is in fact a (G,O(2n))-bundle (see tom Dieck [50]). We have the following isomorphisms of G-vector bundles

$$E \cong P \times_{O(2n)} \mathbb{R}^{2n}, \quad \psi^k E = P \times_{O(2n)} \psi^k \mathbb{R}^{2n}.$$

Hence we obtain from Proposition 11.3.1.

Proposition 11.3.7. Let G _be a compact Lie group and let_ $E \longrightarrow B$ _be an orthogonal_ G-_vector bundle. Then there exist stable_ G-_maps_ $S(E) \longrightarrow S(\psi^k E)$ _if_ k _is odd_ ($S(E \oplus E) \longrightarrow S(\psi^k(E \oplus E))$ _if_ k _is even_) _of fibre-degree dividing a power of_ k.

One actually would like to have an information about the degrees on fixed point sets. By the methods of Quillen [128] one can prove the following equivariant version of the Adams conjecture.

Theorem 11.3.8. There exist stable G-_maps_ $f : S(E) \longrightarrow S(\psi^k E)$ _such that_ f^H _has for all_ $H < G$ _a degree which divides a power of_ k. (k _prime to_ $|G|$).

By the results of section 9 and 10 this is easy to see for bundles with finite structure group.

11.4. Local J-groups at p.

Let G be a finite p group and let $r \in \mathbb{N}$ be an odd generator of the p-adic units (mod $\pm$ 1 if p = 2). Let X be a finite connected G-CW-complex. The main result of this section is

Theorem 11.4.1. The following sequence is exact

$$KO_G(X)_p \xrightarrow{1-\psi^r} KO_G(X)_p \xrightarrow{J} JO_G^{loc}(X)_p.$$

(The map J is the quotient map.)

The proof consists in a sequence of Propositions. Recall definition (2.5) for the next result. Let S be the set of all primes.

Proposition 11.4.2. The canonical quotient map

$$B : JO_G^{loc}(X)_p \longrightarrow JO_{G,\{p\}}(X)_p$$

is an isomorphism.

Proof. Suppose B(E - F) = O. Then we can find stable G-maps $f : SE \longrightarrow SF$ and $g : SF \longrightarrow SE$ of degree prime to p. By a theorem of Adams [2] , we can find a stable map $h:S(kE) \longrightarrow S(kF)$ of degree one, where $(k,p) = 1$. Hence (using induction) there exists a stable G-map $h' : S(kE) \longrightarrow S(kF)$ of degree $p^n = |G|$. Since $(\deg(f), \deg(h')) = 1$ a suitable linear combination of f and h' will yield a stable G-map $v : S(kE) \longrightarrow S(kF)$ of degree 1. The same reasoning can be applied to g, and to fixed point mappings. Hence E-F is zero in $JO_G^{loc}(X)_p$.

We now have to consider fibrewise localizations of sphere bundles in the sense of Sullivan [157] . In order to talk about something definite we use the following construction for such localizations. Let $E \longrightarrow B$ be an orthogonal G-vector bundle and $P \longrightarrow B$ be the associated principal $(G,O(n))$-bundle. Let $O(n)$ act on $\mathbb{R}^n \oplus \mathbb{R}^k$, $k \geq 3$, through the standard action on $\mathbb{R}^n$. Let $S(\mathbb{R}^n \oplus \mathbb{R}^k)_p$ be the p-local sphere obtained from a telescope-construction applied to a diagram

$$S(\mathbb{R}^{n+k}) \xrightarrow[f_1]{} S(\mathbb{R}^{n+k}) \xrightarrow[f_2]{} \dots$$

where the maps f_i are the identity on $S(\mathbb{R}^n)$ in $S(\mathbb{R}^n \oplus \mathbb{R}^k) = S(\mathbb{R}^n) * S(\mathbb{R}^k)$. Then $S(\mathbb{R}^{n+k})_p$ still carries an $O(n)$-action and

$$P \times_{O(n)} S(\mathbb{R}^{n+k})_p$$

is our stable representative for the p-local sphere bundle associated to $E \longrightarrow B$. By abuse of notation we denote this bundle $S(E)_p$. We use the fact that $S(E)_p \longrightarrow B$ is a G-fibration (G-homotopy lifting property for all spaces) if $E \longrightarrow B$ is a numerable bundle.

Proposition 11.4.3. Suppose r is odd and prime to p. Let G be a p-group and X a finite G-CW-complex. Then

$$(1-\psi^r)\, KO_G(X)_p \subset TO_{G,\{p\}}(X)_p.$$

Proof. By Proposition 11.3.7 there exists a stable G-map $f : S(E) \longrightarrow S(\psi^r E)$ of degree prime to p. Since G is a p-group we have $\deg f^H \not\equiv 0 \bmod p$ for all $H < G$. The induced map

$$f_p^H : S(E)_p^H \longrightarrow S(\psi^r E)_p^H$$

is therefore a fibrewise map and a homotopy-equivalence on each fibre. By a theorem of Dold [71] f_p^H is a fibre-homotopy-equivalence. By 8.2.4 f_p is a G-homotopy-equivalence and by the equivariant analogue of Dold [71] therefore a G-fibre-homotopy-equivalence, with inverse $g_p : S(\psi^r E)_p \longrightarrow S(E)_p$ say. Since X is compact the composition

$$S(\psi^r E) \xrightarrow{i} S(\psi^r E)_p \xrightarrow{g_p} S(E)_p,$$

where i, a the canonical map into the telescope, has an image which is contained in a finite piece of the telescope. Therefore we obtain a stable G-map $g : S(\psi^r E) \longrightarrow S(E)$ of degree prime to p. This shows $E \sim_{\{p\}} \psi^r E$.

We remark that the proof above actually shows the following

<u>Proposition 11.4.4.</u> <u>Suppose</u> $f : S(E) \longrightarrow S(F)$ <u>is a stable</u> G-<u>map such that the fibre degrees of</u> f^H <u>divide a power of</u> k. <u>Then there exists a stable</u> G-<u>map</u> $g : S(F) \longrightarrow S(E)$ <u>with the same property</u>.

<u>Proof of Theorem 11.4.1.</u> By Proposition 11.4.2 and 11.4.3 we know that $J \circ (1-\psi^r)$ is zero. Hence we have to show that the induced map

$$Q : KO_G(X)_p/(1-\psi^r) \longrightarrow JO_G^{loc}(X)_p$$

is injective. We use the results of Atiyah-Tall [14] on p-adic λ-rings which we have presented in section 3. We let A_p be the p-adic completion of the abelian group A.

Let $\widetilde{KSO}_G(X)$ be the subgroup of elements of dimension zero. By the results of 11.2, in particular Lemma 11.2.6, we need only show that the map

$$\tilde{Q} : \widetilde{KSO}_G(X)_p/(1-\psi^r) \longrightarrow \widetilde{JSO}_G^{loc}(X)_p$$

is injective.

By Atiyah-Tall [14], III. Proposition 3.1, the p-adic and I(G)-adic topologies on KO_G(Point) coincide. This implies that the p-adic and I(G)-adic topologies on $KSO_G(X)$ coincide, if X is a finite G-CW-complex (use Atiyah-MacDonald [11], 10.13). By the version for orientable vector bundles of the Atiyah-Segal completion theorem [12] one has an isomorphism

$$\alpha : \widetilde{KSO}_G(X)^{\wedge}_p \longrightarrow \widetilde{KSO}(X_G),$$

where $X_G = EG \times_G X$, EG the universal free G-space.

We now consider the following diagram whose ingredients we explain in a moment.

$$\begin{array}{ccccc}
\widetilde{KSO}_G(X)_p/1-\psi^r) & & \xrightarrow{\quad\tilde{Q}\quad} & & \widetilde{JSO}_G^{loc}(X)_p \\
\downarrow i_\Gamma & & (*) & & \\
\widetilde{KSO}_G(X)^{\wedge}_{p,\Gamma} & \xrightarrow[\sigma_{r,\Gamma}]{} & (1+\widetilde{KSO}_G(X)^{\wedge}_p)_\Gamma & & \swarrow \theta_r \\
\cong \downarrow \alpha_\Gamma & & (**) & \alpha_\Gamma \searrow \cong & \\
\widetilde{KSO}(X_G)_\Gamma & & \xrightarrow[\tau_{r,\Gamma}]{} & & (1+\widetilde{KSO})X_G))_\Gamma
\end{array}$$

The index Γ indicates that we factor out the image of $1-\psi^r$. The ring $\widetilde{KSO}_G(X)^{\wedge}_p$ is an orientable p-adic γ-ring; we therefore have the map ρ^{or}_r, as defined in 3.10.7. The map $\sigma_{r,\Gamma}$ is induced by ρ^{or}_r on the quotients. Similarly α_Γ is induced by α and $\tau_{r,\Gamma}$ is defined so as to make (**) commutative. The inclusion $i : \widetilde{KSO}_G(X)_p \longrightarrow \widetilde{KSO}_G(X)^{\wedge}_p$ induces an injective map i_Γ because p-adic completion is exact on finitely-generated Z_p-modules. Since $\sigma_{r,\Gamma}$ is an isomorphism by

3.14.10 we need only demonstrate the existence of a homomorphism Θ_r which makes the diagram commutative.

Suppose $f : S(E) \longrightarrow S(F)$ is a stable G-map of degree zero. Then $EG \times_G S(E)$ and $EG \times_G S(F)$ are fibre homotopy equivalent hence have the same Stiefel-Whitney classes. We therefore may and will assume that they both have a Spin(8n)-structure and hence a K-theory Thom-class. Applying $id \times_G f$ to these Thom-classes and using 3.15 one obtains

$$\tilde{\Theta}_r(EG \times_G E)\ \psi^r(z) = z\ \tilde{\Theta}_r(EG \times_G F)$$

with a suitable $z \in 1 + \widetilde{K}SO(X_G)$ and this yields the desired factorisation.

11.5. Local J-groups away from p.

We now assume that q is a prime different from p and compute the J-groups localized at q.

To begin with let C be a cyclic group and Y a trivial C-space. We can compute $JO_C^{loc}(Y)_q$ as follows.

Since Y is a trivial C-space vector bundles over Y split according to the irreducible C-modules (see Segal [142] , Remark on p. 133). Since C is a cyclic p-group the splitting of vector bundles according to the kernels of the irreducible C-modules is preserved by JO^{loc}-equivalence and by Adams operations. Hence it suffices to discuss that direct summand of $JO_C^{loc}(Y)_q$ which comes from C-vector bundles whose fibre representations only contain faithful C-modules. We claim that forgetting the group action induces an isomorphism of this direct summand with $JO(Y)_q$ (if $q \neq 2$) and with $J(Y)_q$ (if $q = 2$ and C non-trivial). Moreover $JO_C^{loc}(Y)_q$ can be computed as in 11.1.6 in this case. We prove all this.

Let $(r,pq) = 1$. Then there is a stable C map $S(E \otimes V) \longrightarrow S(\psi^r E \otimes \psi^r V)$ of degree t dividing r^n, where V is a faithful C-module and E a bundle with trivial C-action. As in the proof of 11.4.2 we see that there exists a C-map $S(t^i(E \otimes V)) \longrightarrow S(t^i(\psi^r E \otimes \psi^r V))$ for suitable i. Since $(t,q) = 1$ we have that $(1-\psi^r)(E \otimes V)$ is zero in $JO_C^{loc}(X)_q$ (use also 11.4.4).

Now suppose that $E_1 - E_2$ maps to zero in $JO(Y)_q$. For each r generating the q-adic units there exists an F such that $E_1 - E_2 = (1-\psi^r)F$, by the non-equivariant computation of $JO(Y)_q$ which is a special case of the results in 11.4. Hence also $F \otimes V - \psi^r F \otimes V^r$ in $JO_C(Y)_q$. (We can actually work with complex vector bundles, because $J(Y)_q \cong JO(Y)_q$ if $q \neq 2$ and if $q = 2$ then C is not a 2-group and the faithful representations of C are of complex type.) If we choose r such that $V^r = V$ then we see that $E_1 \otimes V - E_2 \otimes V = (1-\psi^r)(F \otimes V)$ maps to zero in $JO_C^{loc}(Y)_q$ is of the form as claimed in 11.1.6. In general if $E_1 - E_2 = (1-\psi^s)F_1$ then $E_1 \otimes V - E_2 \otimes V^s = F_1 \otimes V - \psi^s F_1 \otimes V^s = (F_1 \otimes (V - V^s)) + ((F_1 - \psi^s F_1) \otimes V^s)$ shows that $F_1 \otimes (V - V^s)$ is also contained in the subgroup generated by the $(1 - \psi^{r(i)})$ of 11.1.6. This settles the case of cyclic p-groups C and trivial C-spaces Y.

We now prove 11.1.6 in general for $q \neq p$. By 7.7 we have a natural transformation

$$KO_G(X) \longrightarrow \bigoplus_{(C)} KO_C(X^C)$$

where (C) runs over the conjugacy classes of cyclic subgroups of G. This transformation has a natural splitting which is compatible with the action of the Adams operations. Let $JO'_G(X)$ denote the quotient of $KO_G(X)$ by the subgroups generated by $(1 - \psi^{r(i)})x$ as in 11.1.6. Then we have the diagram

$$\begin{array}{ccc}
KO_G(X)_q & \longrightarrow & \oplus_{(C)} KO_C(X^C)_q \\
\downarrow & & \downarrow \\
JO'_G(X)_q & \xrightarrow{(2)} & \oplus_{(C)} JO'_C(X^C)_q \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (3)} \\
JO_G^{loc}(X)_q & \xrightarrow{(4)} & \oplus_{(C)} JO_C^{loc}(X^C)_q
\end{array}$$

The maps (1) and (3) are surjective by construction. The map (2) is split injective by the splitting theorem just quoted. The map (3) is bijective by the proof above. Hence (1) is also injective hence an isomorphism. This finishes the proof of Theorem 11.1.6.

11.6. Projective modules.

We are going to discuss the difference between JO_G^{loc} and JO_G.

Let E and F be G-vector bundles over X. Let $[S(E),S(F)]$ be the set of G-fibre homotopy classes $S(E) \longrightarrow S(F)$. Fibrewise suspension defines a map $[S(E), S(F)] \longrightarrow [S(E \oplus V), S(F \oplus V)]$. We take the direct limit over such suspension maps and call the limit $\omega_G^0(E,F)$, which is the set of G-homotopy classes of stable maps $S(E) \longrightarrow S(F)$. We list some of the standard properties of this construction.

(11.6.1) $\omega_G^0(E,F)$ is an abelian group and in fact a module over $\omega_G^0(X)$.

(11.6.2) A G-map $f : Y \longrightarrow X$ induces a homomorphism

$$f^* : \omega_G^0(E,F) \longrightarrow \omega_G^0(f^*E, f^*F).$$

(11.6.3) Composition of mappings defines a pairing

$$\omega_G^0(E,F) \times \omega_G^0(F,H) \longrightarrow \omega_G^0(E,H)$$

which is $\omega_G^0(X)$-bilinear.

(11.6.4) Whitney sum defines a pairing

$$\omega_G^0(E_1,F_1) \times \omega_G^0(E_2,F_2) \longrightarrow \omega_G^0(E_1 \oplus E_2, F_1 \oplus F_2)$$

which is ω_G^0-bilinear.

(11.6.5) There are canonical isomorphisms of $\omega_G^0(X)$-modules

$$\omega_G^0(E,E) \cong \omega_G^0(X)$$

$$\omega_G^0(E,E) \cong \omega_G^0(E \oplus F, E \oplus F)$$

<u>Proposition 11.6.7.</u> <u>Suppose</u> $E - F \in TO_G^{loc}(X)$. <u>Then</u> $\omega_G^0(E,F)$ <u>is a projective</u> $\omega_G^0(X)$-<u>module of rank one and</u> $\omega_G^0(F,E)$ <u>is its inverse in the Picard group of</u> $\omega_G^0(X)$. <u>The module is free if and only if</u> $E - F \in TO_G(X)$.

<u>Proof.</u> We have determined the prime ideals q of $\omega_G^0(X)$ in We localize at q and show that $\omega_G^0(E,F)_q$ is a free $\omega_G^0(X)_q$-module of rank one and that $\omega_G^0(E,F) \otimes \omega_G^0(F,E) \longrightarrow \omega_G^0(E \oplus F, F \oplus E) \cong \omega_G^0(X)$ induces an isomorphism after localization at q. But by the definition of $TO_G^{loc}(X)$ we have for a given H a stable G-map $f : S(E) \longrightarrow S(F)$ such that f^H has fibre degree one. Now proceed as in 10.2.6.

From 11.6.7 we obtain an injective homomorphism $pO_X(G) : TO_G^{loc}(X)/TO_G(X) \longrightarrow \mathrm{Pic}\ \omega_G^0(X)$. Note that the source of $pO_X(G)$ is precisely the kernel of $JO_G^{loc} \longrightarrow JO_G$. The Picard group $\omega_G^0(X)$

does not change if we divide out the nilradical of $\omega_G^O(X)$. We have seen that $\omega_G^O(X)/\mathrm{Nil}$ only depends on the orbit category of X. In particular if all the fixed point sets of X are non-empty and connected then we obtain a natural splitting $JO_G(X) \cong JO_G^{loc}(X) \oplus jO(G)$.

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Notation

G	compact Lie group
$H < G$	H closed subgroup of G
$H \triangleleft G$	H closed normal subgroup of G
$NH = N_G H$	normalizer of H in G
$WH = NH/H$	
$H \sim K$	H conjugate to K
(H)	conjugacy class of H
$(H) < (K)$	H subconjugate to K

G-space X	left continuous action of G on X
G_x	isotropy group at $x \in X$
X/G	orbit space of X
$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}$	H-fixed point set of X
$X_{(H)} = \{x \in X \mid (G_x) = (H)\}$	H-orbit bundle of H
$X_H = \{x \in X \mid G_x = X\}$	
$G x_H X$	quotient G x X with respect to $(g,x) \sim (gh, h^{-1}x)$, $h \in H$
$\lvert S \rvert$	cardinality of the set S